Ordinary Differential Equations

Introduction

A differential equation is an equation that involves a function and its derivatives. The simplest kind of function is one input, one output. Let's call the input t and the output u.

The simplest kind of differential equation would only involve the first derivative, so should look something like

$$\frac{du}{dt} = f(t, u)$$

where f is some function (notice we are allowed to use the input variable and the output variable in our equation).

Example 1. *If $f(t, u) = 0$, then we are looking for functions $u(t)$ so that*

$$\frac{du}{dt} = 0$$

The only functions with derivative 0 are constant functions, so

$$u(t) = c$$

for some number c.

Example 2. *Similarly, if $f(t, u) = t$, then we want a function $u(t)$ so that*

$$\frac{du}{dt} = t$$

From Calc 1, you know $u(t) = \frac{1}{2}t^2$ is a function that will do this. In general, any function

$$u(t) = \frac{1}{2}t^2 + c$$

Notice in these examples the right-hand side involves only the input variable, t. In this case, solving the differential equation is the same as doing an (indefinite) integral.

Separable Differential Equations

What if the right-hand side involves u?

Example 3. *If $f(t, u) = u$, then we want a function $u(t)$ so that*

$$\frac{du}{dt} = u$$

You should recall from Calc 1 that the natural exponential function

$$u(t) = e^t$$

has the property that its derivative is equal to itself, so this is a solution to our differential equation.

With some imagination, you might notice that (because of the Constant Multiple Rule), we can multiply the exponential by any number C

$$u(t) = C \cdot e^t$$

and this will also solve the differential equation.

How do we come up with the solutions to this differential equation without just guessing or remembering? Fortunately, this equation has a special property:

Definition 4. *A (first-order) differential equation of the form*

$$\frac{du}{dt} = f(t, u)$$

is called ***separable*** *if the function* f *has the form*

$$f(t, u) = F(t) \cdot G(u)$$

(so the t*-dependence and the* u*-dependence of the right-hand side can be 'separated' into a product).*

Separable differential equations are solvable (in a certain sense, not necessarily with an explicit formula for the solution). The prototype is Example 3, so let's see how to solve it; the idea is to pretend $\frac{du}{dt}$ is a fraction, and separate the variables to each side of the equation. For Example 3:

$$\begin{aligned} \frac{du}{dt} &= u \\ du &= udt \\ \frac{1}{u}dy &= dt \end{aligned}$$

From here, we integrate both sides

$$\begin{aligned} \int \frac{1}{u}du &= \int dt \\ \ln|u| &= t + c \\ e^{\ln|u|} &= e^{t+c} \\ |u| &= e^t \cdot e^c \\ u(t) &= C \cdot e^t \end{aligned}$$

(I like to use the capital C to point out it's slightly different than the constant of integration c, but notice that's where it comes from.)

More Separable Examples

Example 5. *Solve*

$$\frac{dx}{dt} = t \cdot (x^2 + 1)$$

Solution. Separate the variables and integrate:

$$\begin{aligned}
\frac{dx}{dt} &= t(x^2+1) \\
\frac{dx}{x^2+1} &= tdt \\
\int \frac{dx}{x^2+1} &= \int tdt \\
\arctan(x) &= \frac{1}{2}t^2 + c \\
x(t) = \tan\left(\frac{1}{2}t^2 + c\right)
\end{aligned}$$

□

Example 6. *Solve*

$$\frac{du}{dt} = (t+1)u$$

Solution. Same procedure:

$$\begin{aligned}
\frac{du}{dt} &= (t+1)u \\
\frac{du}{u} &= (t+1)dt \\
\int \frac{du}{u} &= \int (t+1)dt \\
\ln(|u|) &= \frac{1}{2}t^2 + t + c \\
u(t) &= Ce^{\frac{1}{2}t^2+t}
\end{aligned}$$

□

Example 7. *Switching notation, solve this differential equation for the unknown function $y(x)$:*

$$y' = \frac{e^x}{y}$$

Solution. As above:

$$\begin{aligned}\frac{dy}{dx} &= \frac{e^x}{y}\\ ydy &= e^x dx\\ \int ydy &= \int e^x dx\\ \frac{1}{2}y^2 &= e^x + c\\ y(x) &= \pm\sqrt{2e^x + k}\end{aligned}$$

(Here I write $k = 2c$ again to emphasize it is not the same c, but it comes from the same place. When I'm working I would probably just write c everywhere, but if you're being careful these constants are different.) □