A (first-order) differential equation

$$\dot{x} = f(x, t)$$

is called **separable** if the function f has the form

$$f(x, t) = F(x) \cdot G(t)$$

(so the x-dependence and the t-dependence can be 'separated' into a product). Separable differential equations are solvable (in a certain sense, not necessarily with an explicit formula for the solution).

The following differential equations are separable:

1)

$$\dot{x} = t \cdot (x^2 + 1)$$

Solution. Writing in Leibniz notation, we rearrange

$$\begin{aligned} \frac{dx}{dt} &= t(x^2+1) \\ \frac{dx}{x^2+1} &= tdt \\ \int \frac{dx}{x^2+1} &= \int tdt \\ \arctan(x) &= \frac{1}{2}t^2 + c \\ x(t) = \tan\left(\frac{1}{2}t^2 + c\right) \end{aligned}$$

Here c is some constant number. If we choose an initial condition $x(0) = x_0$, then we can solve for a specific c and get a formula for a trajectory starting at x_0.

For instance, if we want $x(1) = 1$, then we need

$$\tan\left(\frac{1}{2} + c\right) = 1$$

or

$$c = \frac{\pi}{4} - \frac{1}{2}$$

So our final solution is

$$x(t) = \tan\left(\frac{1}{2}t^2 + \frac{\pi}{4} - \frac{1}{2}\right)$$

□

2)

$$\dot{x} = (t+1)x$$

Solution. Again, we adopt Leibniz notation:

$$\begin{aligned}\frac{dx}{dt} &= (t+1)x \\ \frac{dx}{x} &= (t+1)dt \\ \int \frac{dx}{x} &= \int (t+1)dt \\ \ln(|x|) &= \frac{1}{2}t^2 + t + c \\ x(t) &= Ce^t e^{\frac{1}{2}t^2}\end{aligned}$$

or

$$x(t) = Ce^{\frac{1}{2}t^2+t}$$

□

3) (Switching notation:)

$$y' = \frac{e^x}{y}$$

Solution. As above:

$$\begin{aligned}\frac{dy}{dx} &= \frac{e^x}{y} \\ ydy &= e^x dx \\ \int ydy &= \int e^x dx \\ \frac{1}{2}y^2 &= e^x + c \\ y(x) &= \pm\sqrt{2e^x + k}\end{aligned}$$

□