4 Exponential and Logarithmic Functions

In this chapter we study *exponential functions*. These are functions like $f(x) = 2^x$, where the independent variable is in the exponent. Exponential functions are used in modeling many real-world phenomena, such as the growth of a population, the growth of an investment that earns compound interest, or the decay of a radioactive substance. Once an exponential model has been obtained, we can use the model to predict the size of a population, calculate the amount of an investment, or find the amount of a radioactive substance that remains. The inverse functions of exponential functions are called *logarithmic functions*. With exponential models and logarithmic functions we can answer questions such as these: When will my city be as crowded as the city street pictured here? When will my bank account have a million dollars? When will radiation from a radioactive spill decay to a safe level?

In the *Focus on Modeling* at the end of the chapter we learn how to fit exponential and power curves to data.

4.1 EXPONENTIAL FUNCTIONS

■ Exponential Functions ■ Graphs of Exponential Functions ■ Compound Interest

In this chapter we study a new class of functions called *exponential functions*. For example,

$$f(x) = 2^x$$

is an exponential function (with base 2). Notice how quickly the values of this function increase.

$$f(3) = 2^3 = 8$$
$$f(10) = 2^{10} = 1024$$
$$f(30) = 2^{30} = 1{,}073{,}741{,}824$$

Compare this with the function $g(x) = x^2$, where $g(30) = 30^2 = 900$. The point is that when the variable is in the exponent, even a small change in the variable can cause a dramatic change in the value of the function.

Exponential Functions

To study exponential functions, we must first define what we mean by the exponential expression a^x when x is any real number. In Section 1.2 we defined a^x for $a > 0$ and x a rational number, but we have not yet defined irrational powers. So what is meant by $5^{\sqrt{3}}$ or 2^{π}? To define a^x when x is irrational, we approximate x by rational numbers.

For example, since

$$\sqrt{3} \approx 1.73205\ldots$$

is an irrational number, we successively approximate $a^{\sqrt{3}}$ by the following rational powers:

$$a^{1.7}, a^{1.73}, a^{1.732}, a^{1.7320}, a^{1.73205}, \ldots$$

Intuitively, we can see that these rational powers of a are getting closer and closer to $a^{\sqrt{3}}$. It can be shown by using advanced mathematics that there is exactly one number that these powers approach. We define $a^{\sqrt{3}}$ to be this number.

For example, using a calculator, we find

$$5^{\sqrt{3}} \approx 5^{1.732}$$
$$\approx 16.2411\ldots$$

The more decimal places of $\sqrt{3}$ we use in our calculation, the better our approximation of $5^{\sqrt{3}}$.

The Laws of Exponents are listed on page 14.

It can be proved that the *Laws of Exponents are still true when the exponents are real numbers*.

EXPONENTIAL FUNCTIONS

The **exponential function with base *a*** is defined for all real numbers x by

$$f(x) = a^x$$

where $a > 0$ and $a \neq 1$.

We assume that $a \neq 1$ because the function $f(x) = 1^x = 1$ is just a constant function. Here are some examples of exponential functions:

$$f(x) = 2^x \qquad g(x) = 3^x \qquad h(x) = 10^x$$

Base 2   Base 3   Base 10

EXAMPLE 1 ■ Evaluating Exponential Functions

Let $f(x) = 3^x$, and evaluate the following:

(a) $f(5)$ **(b)** $f(-\frac{2}{3})$

(c) $f(\pi)$ **(d)** $f(\sqrt{2})$

SOLUTION We use a calculator to obtain the values of f.

	Calculator keystrokes	Output
(a) $f(5) = 3^5 = 243$	3 ^ 5 ENTER	243
(b) $f(-\frac{2}{3}) = 3^{-2/3} \approx 0.4807$	3 ^ ((−) 2 ÷ 3) ENTER	0.4807498
(c) $f(\pi) = 3^{\pi} \approx 31.544$	3 ^ π ENTER	31.5442807
(d) $f(\sqrt{2}) = 3^{\sqrt{2}} \approx 4.7288$	3 ^ √ 2 ENTER	4.7288043

Now Try Exercise 7 ■

■ Graphs of Exponential Functions

We first graph exponential functions by plotting points. We will see that the graphs of such functions have an easily recognizable shape.

EXAMPLE 2 ■ Graphing Exponential Functions by Plotting Points

Draw the graph of each function.

(a) $f(x) = 3^x$ **(b)** $g(x) = \left(\frac{1}{3}\right)^x$

SOLUTION We calculate values of $f(x)$ and $g(x)$ and plot points to sketch the graphs in Figure 1.

x	$f(x) = 3^x$	$g(x) = \left(\frac{1}{3}\right)^x$
-3	$\frac{1}{27}$	27
-2	$\frac{1}{9}$	9
-1	$\frac{1}{3}$	3
0	1	1
1	3	$\frac{1}{3}$
2	9	$\frac{1}{9}$
3	27	$\frac{1}{27}$

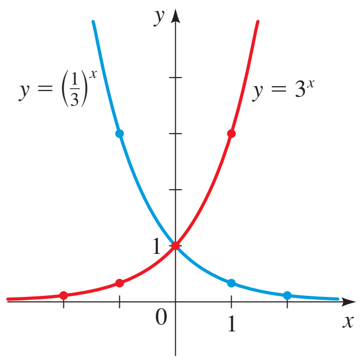


FIGURE 1

Notice that

$$g(x) = \left(\frac{1}{3}\right)^x = \frac{1}{3^x} = 3^{-x} = f(-x)$$

so we could have obtained the graph of g from the graph of f by reflecting in the y-axis.

Reflecting graphs is explained in Section 2.6.

Now Try Exercise 17 ■

Figure 2 shows the graphs of the family of exponential functions $f(x) = a^x$ for various values of the base a. All of these graphs pass through the point $(0, 1)$ because

To see just how quickly $f(x) = 2^x$ increases, let's perform the following thought experiment. Suppose we start with a piece of paper that is a thousandth of an inch thick, and we fold it in half 50 times. Each time we fold the paper, the thickness of the paper stack doubles, so the thickness of the resulting stack would be $2^{50}/1000$ inches. How thick do you think that is? It works out to be more than 17 million miles!

$a^0 = 1$ for $a \neq 0$. You can see from Figure 2 that there are two kinds of exponential functions: If $0 < a < 1$, the exponential function decreases rapidly. If $a > 1$, the function increases rapidly (see the margin note).

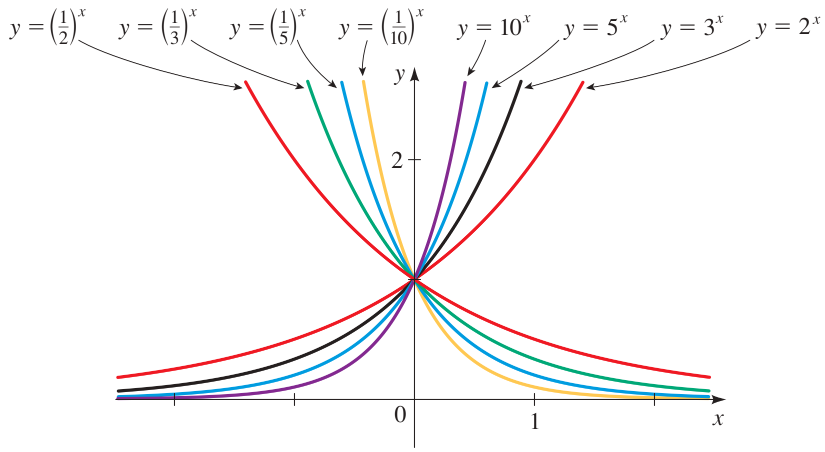


FIGURE 2 A family of exponential functions

See Section 3.6, page 295, where the arrow notation used here is explained.

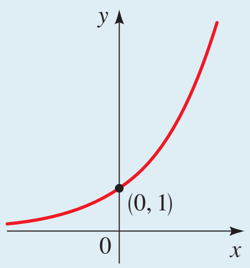
The x-axis is a horizontal asymptote for the exponential function $f(x) = a^x$. This is because when $a > 1$, we have $a^x \to 0$ as $x \to -\infty$, and when $0 < a < 1$, we have $a^x \to 0$ as $x \to \infty$ (see Figure 2). Also, $a^x > 0$ for all $x \in \mathbb{R}$, so the function $f(x) = a^x$ has domain $\mathbb{R}$ and range $(0, \infty)$. These observations are summarized in the following box.

GRAPHS OF EXPONENTIAL FUNCTIONS

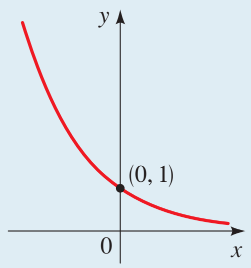
The exponential function

$$f(x) = a^x \qquad a > 0, a \neq 1$$

has domain $\mathbb{R}$ and range $(0, \infty)$. The line $y = 0$ (the x-axis) is a horizontal asymptote of f. The graph of f has one of the following shapes.



$f(x) = a^x$ for $a > 1$

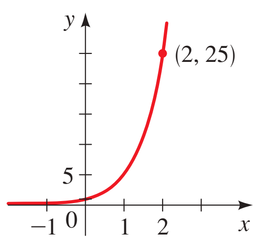


$f(x) = a^x$ for $0 < a < 1$

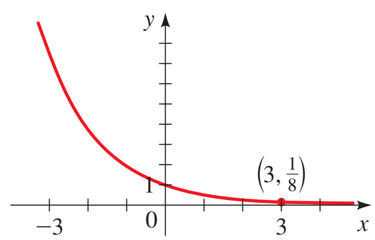
EXAMPLE 3 ■ Identifying Graphs of Exponential Functions

Find the exponential function $f(x) = a^x$ whose graph is given.

(a)



(b)



SOLUTION

(a) Since $f(2) = a^2 = 25$, we see that the base is $a = 5$. So $f(x) = 5^x$.

(b) Since $f(3) = a^3 = \frac{1}{8}$, we see that the base is $a = \frac{1}{2}$. So $f(x) = \left(\frac{1}{2}\right)^x$.

Now Try Exercise 21 ■

In the next example we see how to graph certain functions, not by plotting points, but by taking the basic graphs of the exponential functions in Figure 2 and applying the shifting and reflecting transformations of Section 2.6.

EXAMPLE 4 ■ Transformations of Exponential Functions

Use the graph of $f(x) = 2^x$ to sketch the graph of each function. State the domain, range, and asymptote.

(a) $g(x) = 1 + 2^x$ **(b)** $h(x) = -2^x$ **(c)** $k(x) = 2^{x-1}$

SOLUTION

Shifting and reflecting of graphs are explained in Section 2.6.

(a) To obtain the graph of $g(x) = 1 + 2^x$, we start with the graph of $f(x) = 2^x$ and shift it upward 1 unit to get the graph shown in Figure 3(a). From the graph we see that the domain of g is the set $\mathbb{R}$ of real numbers, the range is the interval $(1, \infty)$, and the line $y = 1$ is a horizontal asymptote.

(b) Again we start with the graph of $f(x) = 2^x$, but here we reflect in the x-axis to get the graph of $h(x) = -2^x$ shown in Figure 3(b). From the graph we see that the domain of h is the set $\mathbb{R}$ of all real numbers, the range is the interval $(-\infty, 0)$, and the line $y = 0$ is a horizontal asymptote.

(c) This time we start with the graph of $f(x) = 2^x$ and shift it to the right by 1 unit to get the graph of $k(x) = 2^{x-1}$ shown in Figure 3(c). From the graph we see that the domain of k is the set $\mathbb{R}$ of all real numbers, the range is the interval $(0, \infty)$, and the line $y = 0$ is a horizontal asymptote.

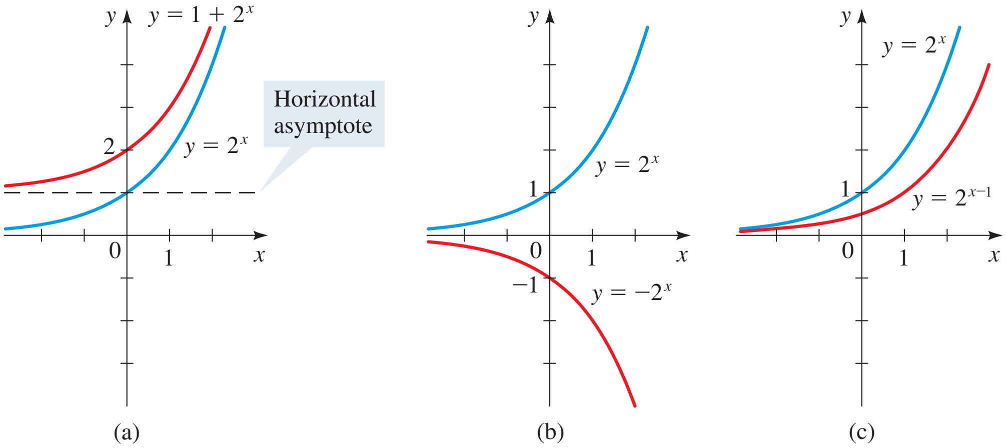


FIGURE 3

Now Try Exercises 27, 29, and 31 ■

EXAMPLE 5 ■ Comparing Exponential and Power Functions

Compare the rates of growth of the exponential function $f(x) = 2^x$ and the power function $g(x) = x^2$ by drawing the graphs of both functions in the following viewing rectangles.

(a) $[0, 3]$ by $[0, 8]$ **(b)** $[0, 6]$ by $[0, 25]$ **(c)** $[0, 20]$ by $[0, 1000]$

SOLUTION

(a) Figure 4(a) shows that the graph of $g(x) = x^2$ catches up with, and becomes higher than, the graph of $f(x) = 2^x$ at $x = 2$.

(b) The larger viewing rectangle in Figure 4(b) shows that the graph of $f(x) = 2^x$ overtakes that of $g(x) = x^2$ when $x = 4$.

(c) Figure 4(c) gives a more global view and shows that when x is large, $f(x) = 2^x$ is much larger than $g(x) = x^2$.

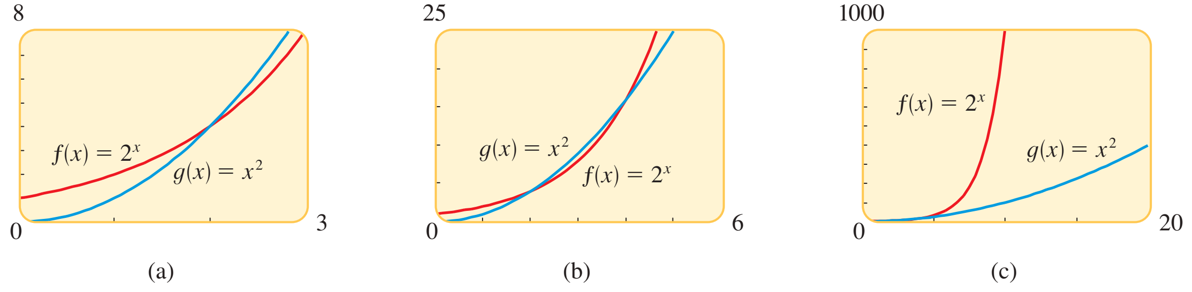


FIGURE 4

Now Try Exercise 45 ■

■ Compound Interest

Exponential functions occur in calculating compound interest. If an amount of money P, called the **principal**, is invested at an interest rate i per time period, then after one time period the interest is Pi, and the amount A of money is

$$A = P + Pi = P(1 + i)$$

If the interest is reinvested, then the new principal is $P(1 + i)$, and the amount after another time period is $A = P(1 + i)(1 + i) = P(1 + i)^2$. Similarly, after a third time period the amount is $A = P(1 + i)^3$. In general, after k periods the amount is

$$A = P(1 + i)^k$$

Notice that this is an exponential function with base $1 + i$.

If the annual interest rate is r and if interest is compounded n times per year, then in each time period the interest rate is $i = r/n$, and there are nt time periods in t years. This leads to the following formula for the amount after t years.

COMPOUND INTEREST

Compound interest is calculated by the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where $A(t)$ = amount after t years

P = principal

r = interest rate per year

n = number of times interest is compounded per year

t = number of years

r is often referred to as the nominal annual interest rate.

EXAMPLE 6 ■ Calculating Compound Interest

A sum of \$1000 is invested at an interest rate of 12% per year. Find the amounts in the account after 3 years if interest is compounded annually, semiannually, quarterly, monthly, and daily.

SOLUTION We use the compound interest formula with $P = \$1000$, $r = 0.12$, and $t = 3$.

Compounding	n	Amount after 3 years
Annual	1	$1000\left(1 + \frac{0.12}{1}\right)^{1(3)} = \$1404.93$
Semiannual	2	$1000\left(1 + \frac{0.12}{2}\right)^{2(3)} = \$1418.52$
Quarterly	4	$1000\left(1 + \frac{0.12}{4}\right)^{4(3)} = \$1425.76$
Monthly	12	$1000\left(1 + \frac{0.12}{12}\right)^{12(3)} = \$1430.77$
Daily	365	$1000\left(1 + \frac{0.12}{365}\right)^{365(3)} = \$1433.24$

Now Try Exercise 57 ■

If an investment earns compound interest, then the **annual percentage yield** (APY) is the *simple* interest rate that yields the same amount at the end of one year.

EXAMPLE 7 ■ Calculating the Annual Percentage Yield

Find the annual percentage yield for an investment that earns interest at a rate of 6% per year, compounded daily.

SOLUTION After one year, a principal P will grow to the amount

$$A = P\left(1 + \frac{0.06}{365}\right)^{365} = P(1.06183)$$

Simple interest is studied in Section 1.7.

The formula for simple interest is

$$A = P(1 + r)$$

Comparing, we see that $1 + r = 1.06183$, so $r = 0.06183$. Thus the annual percentage yield is 6.183%.

Now Try Exercise 63 ■

DISCOVERY PROJECT

So You Want to Be a Millionaire?

In this project we explore how rapidly the values of an exponential function increase by examining some real-world situations. For example, if you save a penny today, two pennies tomorrow, four pennies the next day, and so on, how long do you have to continue saving in this way before you become a millionaire? You can find out the surprising answer to this and other questions by completing this discovery project. You can find the project at **www.stewartmath.com**.

4.1 EXERCISES

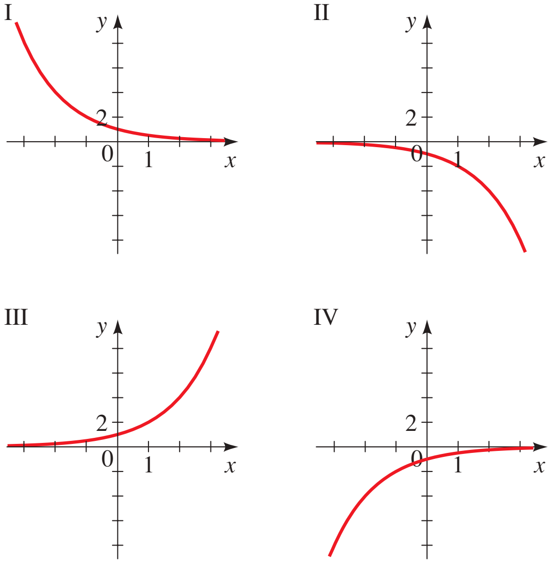
CONCEPTS

1. The function $f(x) = 5^x$ is an exponential function with base ________; $f(-2) =$ ________, $f(0) =$ ________, $f(2) =$ ________, and $f(6) =$ ________.

2. Match the exponential function with one of the graphs labeled I, II, III, or IV, shown below.

(a) $f(x) = 2^x$ **(b)** $f(x) = 2^{-x}$

(c) $f(x) = -2^x$ **(d)** $f(x) = -2^{-x}$



3. (a) To obtain the graph of $g(x) = 2^x - 1$, we start with the graph of $f(x) = 2^x$ and shift it ____________ (upward/downward) 1 unit.

(b) To obtain the graph of $h(x) = 2^{x-1}$, we start with the graph of $f(x) = 2^x$ and shift it to the ____________ (left/right) 1 unit.

4. In the formula $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ for compound interest the letters P, r, n, and t stand for ____________, ____________, ____________, and ____________, respectively, and $A(t)$ stands for ____________. So if \$100 is invested at an interest rate of 6% compounded quarterly, then the amount after 2 years is ____________.

5. The exponential function $f(x) = \left(\frac{1}{2}\right)^x$ has the ______________ asymptote $y =$ ________. This means that as $x \to \infty$, we have $\left(\frac{1}{2}\right)^x \to$ ________.

6. The exponential function $f(x) = \left(\frac{1}{2}\right)^x + 3$ has the ______________ asymptote $y =$ ________. This means that as $x \to \infty$, we have $\left(\frac{1}{2}\right)^x + 3 \to$ ________.

SKILLS

7–10 ■ Evaluating Exponential Functions Use a calculator to evaluate the function at the indicated values. Round your answers to three decimals.

7. $f(x) = 4^x$; $f\left(\frac{1}{2}\right), f(\sqrt{5}), f(-2), f(0.3)$

8. $f(x) = 3^{x-1}$; $f\left(\frac{1}{2}\right), f(2.5), f(-1), f\left(\frac{1}{4}\right)$

9. $g(x) = \left(\frac{1}{3}\right)^{x+1}$; $g\left(\frac{1}{2}\right), g(\sqrt{2}), g(-3.5), g(-1.4)$

10. $g(x) = \left(\frac{4}{3}\right)^{3x}$; $g\left(-\frac{1}{2}\right), g(\sqrt{6}), g(-3), g\left(\frac{4}{3}\right)$

11–16 ■ Graphing Exponential Functions Sketch the graph of the function by making a table of values. Use a calculator if necessary.

11. $f(x) = 2^x$ **12.** $g(x) = 8^x$

13. $f(x) = \left(\frac{1}{3}\right)^x$ **14.** $h(x) = (1.1)^x$

15. $g(x) = 3(1.3)^x$ **16.** $h(x) = 2\left(\frac{1}{4}\right)^x$

17–20 ■ Graphing Exponential Functions Graph both functions on one set of axes.

17. $f(x) = 2^x$ and $g(x) = 2^{-x}$

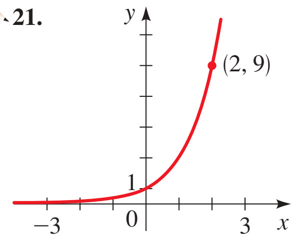
18. $f(x) = 3^{-x}$ and $g(x) = \left(\frac{1}{3}\right)^x$

19. $f(x) = 4^x$ and $g(x) = 7^x$

20. $f(x) = \left(\frac{3}{4}\right)^x$ and $g(x) = 1.5^x$

21–24 ■ Exponential Functions from a Graph Find the exponential function $f(x) = a^x$ whose graph is given.

21.



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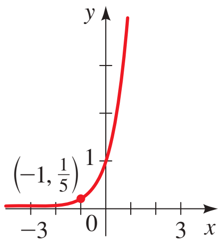

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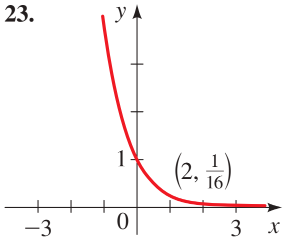
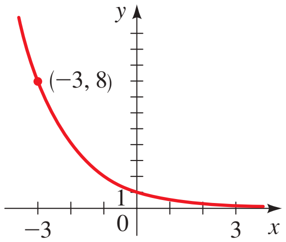
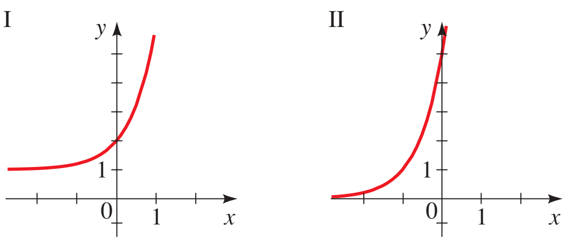


24.



25–26 ■ Exponential Functions from a Graph Match the exponential function with one of the graphs labeled I or II.

25. $f(x) = 5^{x+1}$ **26.** $f(x) = 5^x + 1$



27–40 ■ Graphing Exponential Functions Graph the function, not by plotting points, but by starting from the graphs in Figure 2. State the domain, range, and asymptote.

27. $g(x) = 2^x - 3$ **28.** $h(x) = 4 + \left(\frac{1}{2}\right)^x$

29. $f(x) = -3^x$ **30.** $f(x) = 10^{-x}$

31. $f(x) = 10^{x+3}$ **32.** $g(x) = 2^{x-3}$

33. $y = 5^{-x} + 1$ **34.** $h(x) = 6 - 3^x$

35. $y = 2 - \left(\frac{1}{3}\right)^x$ **36.** $y = 5^{-x} - 3$

37. $h(x) = 2^{x-4} + 1$ **38.** $y = 3 - 10^{x-1}$

39. $g(x) = 1 - 3^{-x}$ **40.** $y = 3 - \left(\frac{1}{5}\right)^x$

41–42 ■ Comparing Exponential Functions In these exercises we compare the graphs of two exponential functions.

41. **(a)** Sketch the graphs of $f(x) = 2^x$ and $g(x) = 3(2^x)$.
(b) How are the graphs related?

42. **(a)** Sketch the graphs of $f(x) = 9^{x/2}$ and $g(x) = 3^x$.
(b) Use the Laws of Exponents to explain the relationship between these graphs.

43–44 ■ Comparing Exponential and Power Functions Compare the graphs of the power function f and exponential function g by evaluating both of them for $x = 0, 1, 2, 3, 4, 6, 8$, and 10. Then draw the graphs of f and g on the same set of axes.

43. $f(x) = x^3$; $g(x) = 3^x$ **44.** $f(x) = x^4$; $g(x) = 4^x$

45–46 ■ Comparing Exponential and Power Functions In these exercises we use a graphing calculator to compare the rates of growth of the graphs of a power function and an exponential function.

45. **(a)** Compare the rates of growth of the functions $f(x) = 2^x$ and $g(x) = x^5$ by drawing the graphs of both functions in the following viewing rectangles.
(i) $[0, 5]$ by $[0, 20]$
(ii) $[0, 25]$ by $[0, 10^7]$
(iii) $[0, 50]$ by $[0, 10^8]$
(b) Find the solutions of the equation $2^x = x^5$, rounded to one decimal place.

46. **(a)** Compare the rates of growth of the functions $f(x) = 3^x$ and $g(x) = x^4$ by drawing the graphs of both functions in the following viewing rectangles:
(i) $[-4, 4]$ by $[0, 20]$
(ii) $[0, 10]$ by $[0, 5000]$
(iii) $[0, 20]$ by $[0, 10^5]$
(b) Find the solutions of the equation $3^x = x^4$, rounded to two decimal places.

SKILLS Plus

47–48 ■ Families of Functions Draw graphs of the given family of functions for $c = 0.25, 0.5, 1, 2, 4$. How are the graphs related?

47. $f(x) = c2^x$ **48.** $f(x) = 2^{cx}$

49–50 ■ Getting Information from a Graph Find, rounded to two decimal places, **(a)** the intervals on which the function is increasing or decreasing and **(b)** the range of the function.

49. $y = 10^{x-x^2}$ **50.** $y = x2^x$

51–52 ■ Difference Quotients These exercises involve a difference quotient for an exponential function.

51. If $f(x) = 10^x$, show that

$$\frac{f(x+h) - f(x)}{h} = 10^x\left(\frac{10^h - 1}{h}\right)$$

52. If $f(x) = 3^{x-1}$, show that

$$\frac{f(x+h) - f(x)}{h} = 3^{x-1}\left(\frac{3^h - 1}{h}\right)$$

APPLICATIONS

53. Bacteria Growth A bacteria culture contains 1500 bacteria initially and doubles every hour.
(a) Find a function N that models the number of bacteria after t hours.
(b) Find the number of bacteria after 24 hours.

54. Mouse Population A certain breed of mouse was introduced onto a small island with an initial population of 320 mice, and scientists estimate that the mouse population is doubling every year.
(a) Find a function N that models the number of mice after t years.
(b) Estimate the mouse population after 8 years.

55–56 ■ Compound Interest An investment of $5000 is deposited into an account in which interest is compounded monthly. Complete the table by filling in the amounts to which the investment grows at the indicated times or interest rates.

55. $r = 4\%$

Time (years)	Amount
1	
2	
3	
4	
5	
6	

56. $t = 5$ years

Rate per year	Amount
1%	
2%	
3%	
4%	
5%	
6%	

57. Compound Interest If $10,000 is invested at an interest rate of 3% per year, compounded semiannually, find the value of the investment after the given number of years.
(a) 5 years **(b)** 10 years **(c)** 15 years

58. Compound Interest If $2500 is invested at an interest rate of 2.5% per year, compounded daily, find the value of the investment after the given number of years.
(a) 2 years **(b)** 3 years **(c)** 6 years

59. Compound Interest If $500 is invested at an interest rate of 3.75% per year, compounded quarterly, find the value of the investment after the given number of years.
(a) 1 year **(b)** 2 years **(c)** 10 years

60. Compound Interest If $4000 is borrowed at a rate of 5.75% interest per year, compounded quarterly, find the amount due at the end of the given number of years.
(a) 4 years **(b)** 6 years **(c)** 8 years

61–62 ■ Present Value The **present value** of a sum of money is the amount that must be invested now, at a given rate of interest, to produce the desired sum at a later date.

61. Find the present value of \$10,000 if interest is paid at a rate of 9% per year, compounded semiannually, for 3 years.

62. Find the present value of \$100,000 if interest is paid at a rate of 8% per year, compounded monthly, for 5 years.

63. Annual Percentage Yield Find the annual percentage yield for an investment that earns 8% per year, compounded monthly.

64. Annual Percentage Yield Find the annual percentage yield for an investment that earns $5\frac{1}{2}\%$ per year, compounded quarterly.

DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

65. DISCUSS ■ DISCOVER: Growth of an Exponential Function Suppose you are offered a job that lasts one month, and you are to be very well paid. Which of the following methods of payment is more profitable for you?

(a) One million dollars at the end of the month

(b) Two cents on the first day of the month, 4 cents on the second day, 8 cents on the third day, and, in general, 2^n cents on the nth day

66. DISCUSS ■ DISCOVER: The Height of the Graph of an Exponential Function Your mathematics instructor asks you to sketch a graph of the exponential function

$$f(x) = 2^x$$

for x between 0 and 40, using a scale of 10 units to one inch. What are the dimensions of the sheet of paper you will need to sketch this graph?

4.2 THE NATURAL EXPONENTIAL FUNCTION

■ The Number e ■ The Natural Exponential Function ■ Continuously Compounded Interest

Any positive number can be used as a base for an exponential function. In this section we study the special base e, which is convenient for applications involving calculus.

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The **Gateway Arch** in St. Louis, Missouri, is shaped in the form of the graph of a combination of exponential functions (*not* a parabola, as it might first appear). Specifically, it is a **catenary**, which is the graph of an equation of the form

$$y = a(e^{bx} + e^{-bx})$$

(see Exercises 17 and 19). This shape was chosen because it is optimal for distributing the internal structural forces of the arch. Chains and cables suspended between two points (for example, the stretches of cable between pairs of telephone poles) hang in the shape of a catenary.

■ The Number e

The number e is defined as the value that $(1 + 1/n)^n$ approaches as n becomes large. (In calculus this idea is made more precise through the concept of a limit.) The table shows the values of the expression $(1 + 1/n)^n$ for increasingly large values of n.

n	$\left(1 + \frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

It appears that, rounded to five decimal places, $e \approx 2.71828$; in fact, the approximate value to 20 decimal places is

$$e \approx 2.71828182845904523536$$

It can be shown that e is an irrational number, so we cannot write its exact value in decimal form.

■ The Natural Exponential Function

The notation e was chosen by Leonhard Euler (see page 63), probably because it is the first letter of the word *exponential*.

The number e is the base for the natural exponential function. Why use such a strange base for an exponential function? It might seem at first that a base such as 10 is easier to work with. We will see, however, that in certain applications the number e is the best

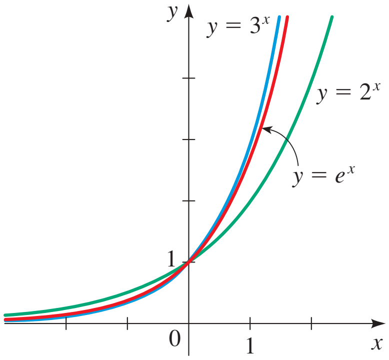


FIGURE 1 Graph of the natural exponential function

possible base. In this section we study how e occurs in the description of compound interest.

THE NATURAL EXPONENTIAL FUNCTION

The **natural exponential function** is the exponential function

$$f(x) = e^x$$

with base e. It is often referred to as *the* exponential function.

Since $2 < e < 3$, the graph of the natural exponential function lies between the graphs of $y = 2^x$ and $y = 3^x$, as shown in Figure 1.

Scientific calculators have a special key for the function $f(x) = e^x$. We use this key in the next example.

EXAMPLE 1 ■ Evaluating the Exponential Function

Evaluate each expression rounded to five decimal places.

(a) e^3 **(b)** $2e^{-0.53}$ **(c)** $e^{4.8}$

SOLUTION We use the e^x key on a calculator to evaluate the exponential function.

(a) $e^3 \approx 20.08554$ **(b)** $2e^{-0.53} \approx 1.17721$ **(c)** $e^{4.8} \approx 121.51042$

Now Try Exercise 3 ■

EXAMPLE 2 ■ Graphing the Exponential Functions

Sketch the graph of each function. State the domain, range, and asymptote.

(a) $f(x) = e^{-x}$ **(b)** $g(x) = 3e^{0.5x}$

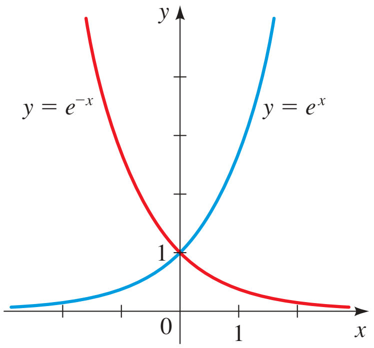


FIGURE 2

SOLUTION

(a) We start with the graph of $y = e^x$ and reflect in the y-axis to obtain the graph of $y = e^{-x}$ as in Figure 2. From the graph we see that the domain of f is the set $\mathbb{R}$ of all real numbers, the range is the interval $(0, \infty)$, and the line $y = 0$ is a horizontal asymptote.

(b) We calculate several values, plot the resulting points, then connect the points with a smooth curve. The graph is shown in Figure 3. From the graph we see that the domain of g is the set $\mathbb{R}$ of all real numbers, the range is the interval $(0, \infty)$, and the line $y = 0$ is a horizontal asymptote.

x	$f(x) = 3e^{0.5x}$
−3	0.67
−2	1.10
−1	1.82
0	3.00
1	4.95
2	8.15
3	13.45

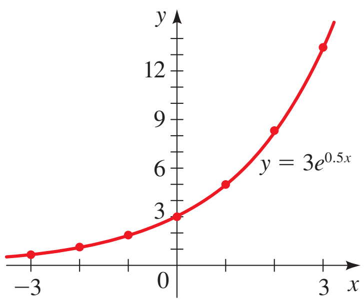


FIGURE 3

Now Try Exercises 5 and 7 ■

EXAMPLE 3 ■ An Exponential Model for the Spread of a Virus

An infectious disease begins to spread in a small city of population 10,000. After t days, the number of people who have succumbed to the virus is modeled by the function

$$v(t) = \frac{10{,}000}{5 + 1245e^{-0.97t}}$$

(a) How many infected people are there initially (at time $t = 0$)?

(b) Find the number of infected people after one day, two days, and five days.

(c) Graph the function v, and describe its behavior.

SOLUTION

(a) Since $v(0) = 10{,}000/(5 + 1245e^0) = 10{,}000/1250 = 8$, we conclude that 8 people initially have the disease.

(b) Using a calculator, we evaluate $v(1)$, $v(2)$, and $v(5)$ and then round off to obtain the following values.

Days	Infected people
1	21
2	54
5	678

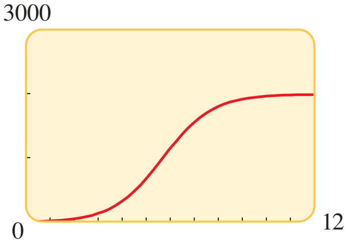


FIGURE 4

$$v(t) = \frac{10{,}000}{5 + 1245e^{-0.97t}}$$

(c) From the graph in Figure 4 we see that the number of infected people first rises slowly, then rises quickly between day 3 and day 8, and then levels off when about 2000 people are infected.

Now Try Exercise 27 ■

The graph in Figure 4 is called a *logistic curve* or a *logistic growth model*. Curves like it occur frequently in the study of population growth. (See Exercises 27–30.)

■ Continuously Compounded Interest

In Example 6 of Section 4.1 we saw that the interest paid increases as the number of compounding periods n increases. Let's see what happens as n increases indefinitely. If we let $m = n/r$, then

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} = P\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}$$

Recall that as m becomes large, the quantity $(1 + 1/m)^m$ approaches the number e. Thus the amount approaches $A = Pe^{rt}$. This expression gives the amount when the interest is compounded at "every instant."

CONTINUOUSLY COMPOUNDED INTEREST

Continuously compounded interest is calculated by the formula

$$A(t) = Pe^{rt}$$

where $A(t)$ = amount after t years
P = principal
r = interest rate per year
t = number of years

EXAMPLE 4 ■ Calculating Continuously Compounded Interest

Find the amount after 3 years if \$1000 is invested at an interest rate of 12% per year, compounded continuously.

SOLUTION We use the formula for continuously compounded interest with $P = \$1000$, $r = 0.12$, and $t = 3$ to get

$$A(3) = 1000e^{(0.12)3} = 1000e^{0.36} = \$1433.33$$

Compare this amount with the amounts in Example 6 of Section 4.1.

Now Try Exercise 33 ■

4.2 EXERCISES

CONCEPTS

1. The function $f(x) = e^x$ is called the ________ exponential function. The number e is approximately equal to ________.

2. In the formula $A(t) = Pe^{rt}$ for continuously compound interest, the letters P, r, and t stand for ________, ________, and ________, respectively, and $A(t)$ stands for ________. So if \$100 is invested at an interest rate of 6% compounded continuously, then the amount after 2 years is ________.

SKILLS

3–4 ■ Evaluating Exponential Functions Use a calculator to evaluate the function at the indicated values. Round your answers to three decimals.

3. $h(x) = e^x$; $h(1), h(\pi), h(-3), h(\sqrt{2})$

4. $h(x) = e^{-3x}$; $h(\frac{1}{3}), h(1.5), h(-1), h(-\pi)$

5–6 ■ Graphing Exponential Functions Complete the table of values, rounded to two decimal places, and sketch a graph of the function.

5.

x	$f(x) = 1.5e^x$
−2	
−1	
−0.5	
0	
0.5	
1	
2	

6.

x	$f(x) = 4e^{-x/3}$
−3	
−2	
−1	
0	
1	
2	
3	

7–16 ■ Graphing Exponential Functions Graph the function, not by plotting points, but by starting from the graph of $y = e^x$ in Figure 1. State the domain, range, and asymptote.

7. $g(x) = 2 + e^x$

8. $h(x) = e^{-x} - 3$

9. $f(x) = -e^x$

10. $y = 1 - e^x$

11. $y = e^{-x} - 1$

12. $f(x) = -e^{-x}$

13. $f(x) = e^{x-2}$

14. $y = e^{x-3} + 4$

15. $h(x) = e^{x+1} - 3$

16. $g(x) = -e^{x-1} - 2$

SKILLS Plus

17. Hyperbolic Cosine Function The *hyperbolic cosine function* is defined by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

(a) Sketch the graphs of the functions $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ on the same axes, and use graphical addition (see Section 2.7) to sketch the graph of $y = \cosh(x)$.

(b) Use the definition to show that $\cosh(-x) = \cosh(x)$.

18. Hyperbolic Sine Function The *hyperbolic sine function* is defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

(a) Sketch the graph of this function using graphical addition as in Exercise 17.

(b) Use the definition to show that $\sinh(-x) = -\sinh(x)$

19. Families of Functions

(a) Draw the graphs of the family of functions

$$f(x) = \frac{a}{2}(e^{x/a} + e^{-x/a})$$

for $a = 0.5$, 1, 1.5, and 2.

(b) How does a larger value of a affect the graph?

20. The Definition of *e* Illustrate the definition of the number e by graphing the curve $y = (1 + 1/x)^x$ and the line $y = e$ on the same screen, using the viewing rectangle $[0, 40]$ by $[0, 4]$.

21–22 ■ Local Extrema Find the local maximum and minimum values of the function and the value of x at which each occurs. State each answer rounded to two decimal places.

21. $g(x) = x^x$, $x > 0$

22. $g(x) = e^x + e^{-2x}$

APPLICATIONS

23. Medical Drugs When a certain medical drug is administered to a patient, the number of milligrams remaining in the patient's bloodstream after t hours is modeled by

$$D(t) = 50e^{-0.2t}$$

How many milligrams of the drug remain in the patient's bloodstream after 3 hours?

24. Radioactive Decay A radioactive substance decays in such a way that the amount of mass remaining after t days is given by the function

$$m(t) = 13e^{-0.015t}$$

where $m(t)$ is measured in kilograms.

(a) Find the mass at time $t = 0$.

(b) How much of the mass remains after 45 days?

25. Sky Diving A sky diver jumps from a reasonable height above the ground. The air resistance she experiences is proportional to her velocity, and the constant of proportionality is 0.2. It can be shown that the downward velocity of the sky diver at time t is given by

$$v(t) = 180(1 - e^{-0.2t})$$

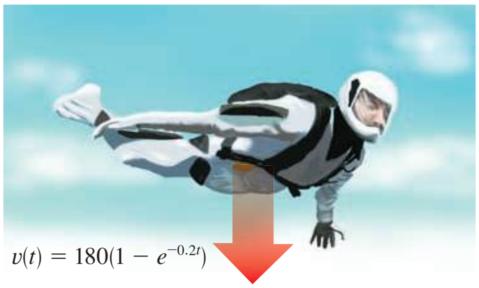
where t is measured in seconds (s) and $v(t)$ is measured in feet per second (ft/s).

(a) Find the initial velocity of the sky diver.

(b) Find the velocity after 5 s and after 10 s.

(c) Draw a graph of the velocity function $v(t)$.

(d) The maximum velocity of a falling object with wind resistance is called its *terminal velocity*. From the graph in part (c) find the terminal velocity of this sky diver.



26. Mixtures and Concentrations A 50-gal barrel is filled completely with pure water. Salt water with a concentration of 0.3 lb/gal is then pumped into the barrel, and the resulting mixture overflows at the same rate. The amount of salt in the barrel at time t is given by

$$Q(t) = 15(1 - e^{-0.04t})$$

where t is measured in minutes and $Q(t)$ is measured in pounds.

(a) How much salt is in the barrel after 5 min?

(b) How much salt is in the barrel after 10 min?

(c) Draw a graph of the function $Q(t)$.

(d) Use the graph in part (c) to determine the value that the amount of salt in the barrel approaches as t becomes large. Is this what you would expect?

$Q(t) = 15(1 - e^{-0.04t})$

27. Logistic Growth Animal populations are not capable of unrestricted growth because of limited habitat and food supplies. Under such conditions the population follows a *logistic growth model*:

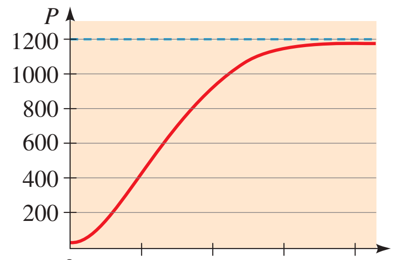
$$P(t) = \frac{d}{1 + ke^{-ct}}$$

where c, d, and k are positive constants. For a certain fish population in a small pond $d = 1200$, $k = 11$, $c = 0.2$, and t is measured in years. The fish were introduced into the pond at time $t = 0$.

(a) How many fish were originally put in the pond?

(b) Find the population after 10, 20, and 30 years.

(c) Evaluate $P(t)$ for large values of t. What value does the population approach as $t \to \infty$? Does the graph shown confirm your calculations?



28. Bird Population The population of a certain species of bird is limited by the type of habitat required for nesting. The population behaves according to the logistic growth model

$$n(t) = \frac{5600}{0.5 + 27.5e^{-0.044t}}$$

where t is measured in years.

(a) Find the initial bird population.

(b) Draw a graph of the function $n(t)$.

(c) What size does the population approach as time goes on?

29. World Population The relative growth rate of world population has been decreasing steadily in recent years. On the basis of this, some population models predict that world population will eventually stabilize at a level that the planet can support. One such logistic model is

$$P(t) = \frac{73.2}{6.1 + 5.9e^{-0.02t}}$$

where $t = 0$ is the year 2000 and population is measured in billions.

(a) What world population does this model predict for the year 2200? For 2300?

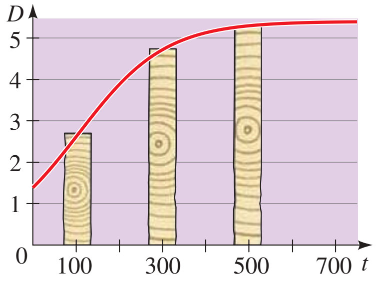
(b) Sketch a graph of the function P for the years 2000 to 2500.

(c) According to this model, what size does the world population seem to approach as time goes on?

30. Tree Diameter For a certain type of tree the diameter D (in feet) depends on the tree's age t (in years) according to the logistic growth model

$$D(t) = \frac{5.4}{1 + 2.9e^{-0.01t}}$$

Find the diameter of a 20-year-old tree.



31–32 ■ Compound Interest An investment of \$7000 is deposited into an account in which interest is compounded continuously. Complete the table by filling in the amounts to which the investment grows at the indicated times or interest rates.

31. $r = 3\%$

Time (years)	Amount
1	
2	
3	
4	
5	
6	

32. $t = 10$ years

Rate per year	Amount
1%	
2%	
3%	
4%	
5%	
6%	

33. Compound Interest If \$2000 is invested at an interest rate of 3.5% per year, compounded continuously, find the value of the investment after the given number of years.

(a) 2 years

(b) 4 years

(c) 12 years

34. Compound Interest If \$3500 is invested at an interest rate of 6.25% per year, compounded continuously, find the value of the investment after the given number of years.

(a) 3 years

(b) 6 years

(c) 9 years

35. Compound Interest If \$600 is invested at an interest rate of 2.5% per year, find the amount of the investment at the end of 10 years for the following compounding methods.

(a) Annually

(b) Semiannually

(c) Quarterly

(d) Continuously

36. Compound Interest If \$8000 is invested in an account for which interest is compounded continuously, find the amount of the investment at the end of 12 years for the following interest rates.

(a) 2%

(b) 3%

(c) 4.5%

(d) 7%

37. Compound Interest Which of the given interest rates and compounding periods would provide the best investment?

(a) $2\frac{1}{2}\%$ per year, compounded semiannually

(b) $2\frac{1}{4}\%$ per year, compounded monthly

(c) 2% per year, compounded continuously

38. Compound Interest Which of the given interest rates and compounding periods would provide the better investment?

(a) $5\frac{1}{8}\%$ per year, compounded semiannually

(b) 5% per year, compounded continuously

39. Investment A sum of \$5000 is invested at an interest rate of 9% per year, compounded continuously.

(a) Find the value $A(t)$ of the investment after t years.

(b) Draw a graph of $A(t)$.

(c) Use the graph of $A(t)$ to determine when this investment will amount to \$25,000.

4.3 LOGARITHMIC FUNCTIONS

■ Logarithmic Functions ■ Graphs of Logarithmic Functions ■ Common Logarithms ■ Natural Logarithms

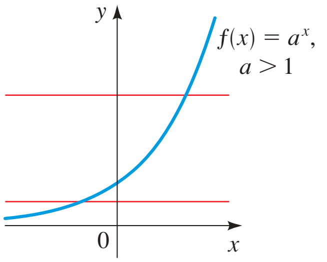


FIGURE 1 $f(x) = a^x$ is one-to-one.

In this section we study the inverses of exponential functions.

■ Logarithmic Functions

Every exponential function $f(x) = a^x$, with $a > 0$ and $a \neq 1$, is a one-to-one function by the Horizontal Line Test (see Figure 1 for the case $a > 1$) and therefore has an inverse function. The inverse function f^{-1} is called the *logarithmic function with base a* and is denoted by $\log_a$. Recall from Section 2.8 that f^{-1} is defined by

$$f^{-1}(x) = y \quad \Leftrightarrow \quad f(y) = x$$

This leads to the following definition of the logarithmic function.

DEFINITION OF THE LOGARITHMIC FUNCTION

Let a be a positive number with $a \neq 1$. The **logarithmic function with base *a***, denoted by $\mathbf{log}_a$, is defined by

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

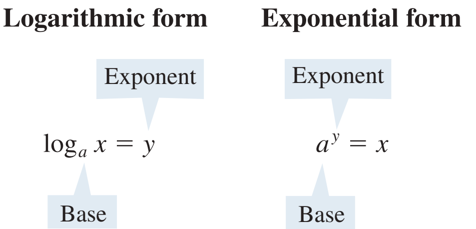
So $\log_a x$ is the *exponent* to which the base a must be raised to give x.

We read $\log_a x = y$ as "log base a of x is y."

By tradition the name of the logarithmic function is $\log_a$, not just a single letter. Also, we usually omit the parentheses in the function notation and write

$$\log_a(x) = \log_a x$$

When we use the definition of logarithms to switch back and forth between the **logarithmic form** $\log_a x = y$ and the **exponential form** $a^y = x$, it is helpful to notice that, in both forms, the base is the same.



EXAMPLE 1 ■ Logarithmic and Exponential Forms

The logarithmic and exponential forms are equivalent equations: If one is true, then so is the other. So we can switch from one form to the other as in the following illustrations.

Logarithmic form	Exponential form
$\log_{10} 100{,}000 = 5$	$10^5 = 100{,}000$
$\log_2 8 = 3$	$2^3 = 8$
$\log_2\left(\frac{1}{8}\right) = -3$	$2^{-3} = \frac{1}{8}$
$\log_5 s = r$	$5^r = s$

Now Try Exercise 7 ■

x	$\log_{10} x$
10^4	4
10^3	3
10^2	2
10	1
1	0
10^{-1}	-1
10^{-2}	-2
10^{-3}	-3
10^{-4}	-4

It is important to understand that $\log_a x$ is an *exponent*. For example, the numbers in the right-hand column of the table in the margin are the logarithms (base 10) of the numbers in the left-hand column. This is the case for all bases, as the following example illustrates.

EXAMPLE 2 ■ Evaluating Logarithms

(a) $\log_{10} 1000 = 3$ because $10^3 = 1000$

(b) $\log_2 32 = 5$ because $2^5 = 32$

(c) $\log_{10} 0.1 = -1$ because $10^{-1} = 0.1$

(d) $\log_{16} 4 = \frac{1}{2}$ because $16^{1/2} = 4$

Now Try Exercises 9 and 11

Inverse Function Property:

$$f^{-1}(f(x)) = x$$
$$f(f^{-1}(x)) = x$$

When we apply the Inverse Function Property described on page 222 to $f(x) = a^x$ and $f^{-1}(x) = \log_a x$, we get

$$\log_a(a^x) = x \qquad x \in \mathbb{R}$$
$$a^{\log_a x} = x \qquad x > 0$$

We list these and other properties of logarithms discussed in this section.

PROPERTIES OF LOGARITHMS

Property	Reason
1. $\log_a 1 = 0$	We must raise a to the power 0 to get 1.
2. $\log_a a = 1$	We must raise a to the power 1 to get a.
3. $\log_a a^x = x$	We must raise a to the power x to get a^x.
4. $a^{\log_a x} = x$	$\log_a x$ is the power to which a must be raised to get x.

EXAMPLE 3 ■ Applying Properties of Logarithms

We illustrate the properties of logarithms when the base is 5.

$\log_5 1 = 0$ Property 1 $\qquad \log_5 5 = 1$ Property 2

$\log_5 5^8 = 8$ Property 3 $\qquad 5^{\log_5 12} = 12$ Property 4

Now Try Exercises 25 and 31

■ Graphs of Logarithmic Functions

Recall that if a one-to-one function f has domain A and range B, then its inverse function f^{-1} has domain B and range A. Since the exponential function $f(x) = a^x$ with $a \neq 1$ has domain $\mathbb{R}$ and range $(0, \infty)$, we conclude that its inverse function, $f^{-1}(x) = \log_a x$, has domain $(0, \infty)$ and range $\mathbb{R}$.

The graph of $f^{-1}(x) = \log_a x$ is obtained by reflecting the graph of $f(x) = a^x$ in the line $y = x$. Figure 2 shows the case $a > 1$. The fact that $y = a^x$ (for $a > 1$) is a very rapidly increasing function for $x > 0$ implies that $y = \log_a x$ is a very slowly increasing function for $x > 1$ (see Exercise 102).

Since $\log_a 1 = 0$, the x-intercept of the function $y = \log_a x$ is 1. The y-axis is a vertical asymptote of $y = \log_a x$ because $\log_a x \to -\infty$ as $x \to 0^+$.

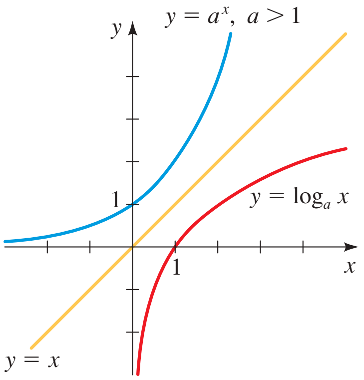


FIGURE 2 Graph of the logarithmic function $f(x) = \log_a x$

EXAMPLE 4 ■ Graphing a Logarithmic Function by Plotting Points

Sketch the graph of $f(x) = \log_2 x$.

SOLUTION To make a table of values, we choose the x-values to be powers of 2 so that we can easily find their logarithms. We plot these points and connect them with a smooth curve as in Figure 3.

x	$\log_2 x$
2^3	3
2^2	2
2	1
1	0
2^{-1}	-1
2^{-2}	-2
2^{-3}	-3
2^{-4}	-4

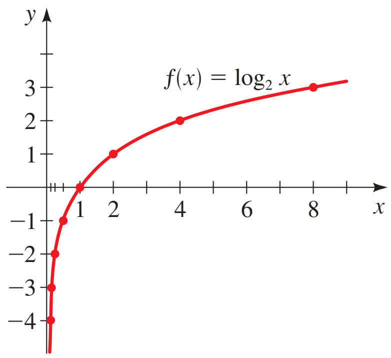


FIGURE 3

Now Try Exercise 49 ■

Figure 4 shows the graphs of the family of logarithmic functions with bases 2, 3, 5, and 10. These graphs are drawn by reflecting the graphs of $y = 2^x$, $y = 3^x$, $y = 5^x$, and $y = 10^x$ (see Figure 2 in Section 4.1) in the line $y = x$. We can also plot points as an aid to sketching these graphs, as illustrated in Example 4.

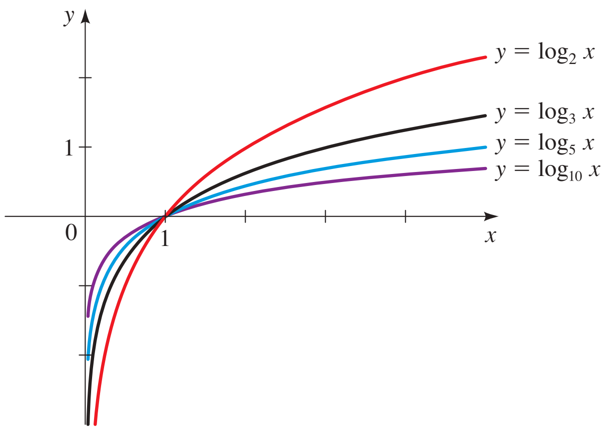


FIGURE 4 A family of logarithmic functions

In the next two examples we graph logarithmic functions by starting with the basic graphs in Figure 4 and using the transformations of Section 2.6.

EXAMPLE 5 ■ Reflecting Graphs of Logarithmic Functions

Sketch the graph of each function. State the domain, range, and asymptote.

(a) $g(x) = -\log_2 x$ **(b)** $h(x) = \log_2(-x)$

SOLUTION

(a) We start with the graph of $f(x) = \log_2 x$ and reflect in the x-axis to get the graph of $g(x) = -\log_2 x$ in Figure 5(a). From the graph we see that the domain of g is $(0, \infty)$, the range is the set $\mathbb{R}$ of all real numbers, and the line $x = 0$ is a vertical asymptote.

Mathematics in the Modern World

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Law Enforcement

Mathematics aids law enforcement in numerous and surprising ways, from the reconstruction of bullet trajectories to determining the time of death to calculating the probability that a DNA sample is from a particular person. One interesting use is in the search for missing persons. A person who has been missing for several years might look quite different from his or her most recent available photograph. This is particularly true if the missing person is a child. Have you ever wondered what you will look like 5, 10, or 15 years from now?

Researchers have found that different parts of the body grow at different rates. For example, you have no doubt noticed that a baby's head is much larger relative to its body than an adult's. As another example, the ratio of arm length to height is $\frac{1}{3}$ in a child but about $\frac{2}{5}$ in an adult. By collecting data and analyzing the graphs, researchers are able to determine the functions that model growth. As in all growth phenomena, exponential and logarithmic functions play a crucial role. For instance, the formula that relates arm length l to height h is $l = ae^{kh}$ where a and k are constants. By studying various physical characteristics of a person, mathematical biologists model each characteristic by a function that describes how it changes over time. Models of facial characteristics can be programmed into a computer to give a picture of how a person's appearance changes over time. These pictures aid law enforcement agencies in locating missing persons.

(b) We start with the graph of $f(x) = \log_2 x$ and reflect in the y-axis to get the graph of $h(x) = \log_2(-x)$ in Figure 5(b). From the graph we see that the domain of h is $(-\infty, 0)$, the range is the set $\mathbb{R}$ of all real numbers, and the line $x = 0$ is a vertical asymptote.

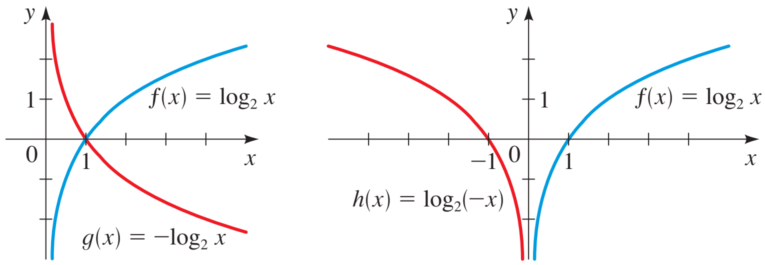


FIGURE 5 (a) (b)

Now Try Exercise 61 ■

EXAMPLE 6 ■ Shifting Graphs of Logarithmic Functions

Sketch the graph of each function. State the domain, range, and asymptote.

(a) $g(x) = 2 + \log_5 x$ **(b)** $h(x) = \log_{10}(x - 3)$

SOLUTION

(a) The graph of g is obtained from the graph of $f(x) = \log_5 x$ (Figure 4) by shifting upward 2 units, as shown in Figure 6. From the graph we see that the domain of g is $(0, \infty)$, the range is the set $\mathbb{R}$ of all real numbers, and the line $x = 0$ is a vertical asymptote.

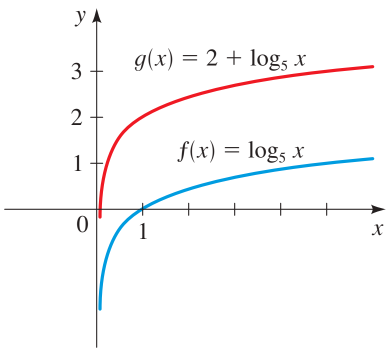


FIGURE 6

(b) The graph of h is obtained from the graph of $f(x) = \log_{10} x$ (Figure 4) by shifting to the right 3 units, as shown in Figure 7. From the graph we see that the domain of h is $(3, \infty)$, the range is the set $\mathbb{R}$ of all real numbers, and the line $x = 3$ is a vertical asymptote.

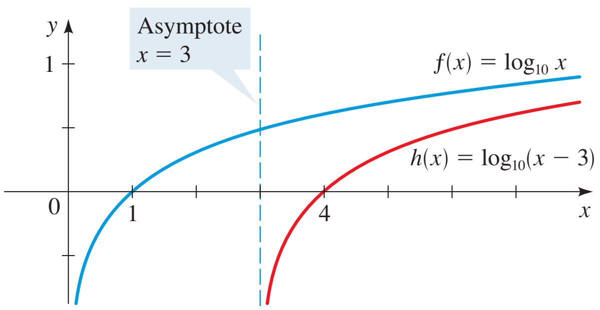
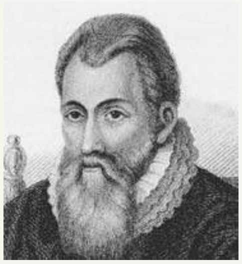


FIGURE 7

Now Try Exercises 63 and 67 ■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JOHN NAPIER (1550–1617) was a Scottish landowner for whom mathematics was a hobby. We know him today because of his key invention: logarithms, which he published in 1614 under the title *A Description of the Marvelous Rule of Logarithms*. In Napier's time, logarithms were used exclusively for simplifying complicated calculations. For example, to multiply two large numbers, we would write them as powers of 10. The exponents are simply the logarithms of the numbers. For instance,

$$\begin{aligned} &4532 \times 57783 \\ &\approx 10^{3.65629} \times 10^{4.76180} \\ &= 10^{8.41809} \\ &\approx 261{,}872{,}564 \end{aligned}$$

The idea is that multiplying powers of 10 is easy (we simply add their exponents). Napier produced extensive tables giving the logarithms (or exponents) of numbers. Since the advent of calculators and computers, logarithms are no longer used for this purpose. The logarithmic functions, however, have found many applications, some of which are described in this chapter.

Napier wrote on many topics. One of his most colorful works is a book entitled *A Plaine Discovery of the Whole Revelation of Saint John*, in which he predicted that the world would end in the year 1700.

Common Logarithms

We now study logarithms with base 10.

COMMON LOGARITHM

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base:

$$\log x = \log_{10} x$$

From the definition of logarithms we can easily find that

$$\log 10 = 1 \qquad \text{and} \qquad \log 100 = 2$$

But how do we find log 50? We need to find the exponent y such that $10^y = 50$. Clearly, 1 is too small and 2 is too large. So

$$1 < \log 50 < 2$$

To get a better approximation, we can experiment to find a power of 10 closer to 50. Fortunately, scientific calculators are equipped with a LOG key that directly gives values of common logarithms.

EXAMPLE 7 ■ Evaluating Common Logarithms

Use a calculator to find appropriate values of $f(x) = \log x$, and use the values to sketch the graph.

SOLUTION We make a table of values, using a calculator to evaluate the function at those values of x that are not powers of 10. We plot those points and connect them by a smooth curve as in Figure 8.

x	$\log x$
0.01	−2
0.1	−1
0.5	−0.301
1	0
4	0.602
5	0.699
10	1

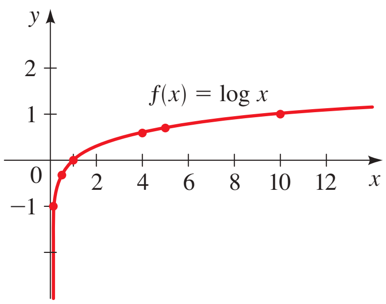


FIGURE 8

Now Try Exercise 51 ■

Scientists model human response to stimuli (such as sound, light, or pressure) using logarithmic functions. For example, the intensity of a sound must be increased manyfold before we "feel" that the loudness has simply doubled. The psychologist Gustav Fechner formulated the law as

$$S = k \log\left(\frac{I}{I_0}\right)$$

where S is the subjective intensity of the stimulus, I is the physical intensity of the stimulus, I_0 stands for the threshold physical intensity, and k is a constant that is different for each sensory stimulus.

Human response to sound and light intensity is logarithmic.

EXAMPLE 8 ■ Common Logarithms and Sound

We study the decibel scale in more detail in Section 4.7.

The perception of the loudness B (in decibels, dB) of a sound with physical intensity I (in W/m^2) is given by

$$B = 10 \log\left(\frac{I}{I_0}\right)$$

where I_0 is the physical intensity of a barely audible sound. Find the decibel level (loudness) of a sound whose physical intensity I is 100 times that of I_0.

SOLUTION We find the decibel level B by using the fact that $I = 100I_0$.

$$\begin{aligned} B &= 10 \log\left(\frac{I}{I_0}\right) && \text{Definition of } B \\ &= 10 \log\left(\frac{100I_0}{I_0}\right) && I = 100I_0 \\ &= 10 \log 100 && \text{Cancel } I_0 \\ &= 10 \cdot 2 = 20 && \text{Definition of log} \end{aligned}$$

The loudness of the sound is 20 dB.

Now Try Exercise 97

Natural Logarithms

Of all possible bases a for logarithms, it turns out that the most convenient choice for the purposes of calculus is the number e, which we defined in Section 4.2.

The notation ln is an abbreviation for the Latin name *logarithmus naturalis*.

NATURAL LOGARITHM

The logarithm with base e is called the **natural logarithm** and is denoted by **ln**:

$$\ln x = \log_e x$$

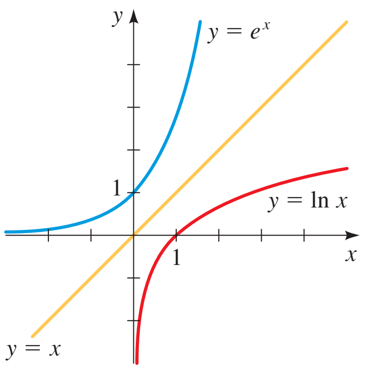


FIGURE 9 Graph of the natural logarithmic function

The natural logarithmic function $y = \ln x$ is the inverse function of the natural exponential function $y = e^x$. Both functions are graphed in Figure 9. By the definition of inverse functions we have

$$\ln x = y \quad \Leftrightarrow \quad e^y = x$$

If we substitute $a = e$ and write "ln" for "$\log_e$" in the properties of logarithms mentioned earlier, we obtain the following properties of natural logarithms.

PROPERTIES OF NATURAL LOGARITHMS

Property	Reason
1. $\ln 1 = 0$	We must raise e to the power 0 to get 1.
2. $\ln e = 1$	We must raise e to the power 1 to get e.
3. $\ln e^x = x$	We must raise e to the power x to get e^x.
4. $e^{\ln x} = x$	$\ln x$ is the power to which e must be raised to get x.

Calculators are equipped with an LN key that directly gives the values of natural logarithms.

EXAMPLE 9 ■ Evaluating the Natural Logarithm Function

(a) $\ln e^8 = 8$ — Definition of natural logarithm

(b) $\ln\left(\dfrac{1}{e^2}\right) = \ln e^{-2} = -2$ — Definition of natural logarithm

(c) $\ln 5 \approx 1.609$ — Use LN key on calculator

Now Try Exercise 47 ■

EXAMPLE 10 ■ Finding the Domain of a Logarithmic Function

Find the domain of the function $f(x) = \ln(4 - x^2)$.

SOLUTION As with any logarithmic function, $\ln x$ is defined when $x > 0$. Thus the domain of f is

$$\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\}$$
$$= \{x \mid -2 < x < 2\} = (-2, 2)$$

Now Try Exercise 73 ■

EXAMPLE 11 ■ Drawing the Graph of a Logarithmic Function

Draw the graph of the function $y = x \ln(4 - x^2)$, and use it to find the asymptotes and local maximum and minimum values.

SOLUTION As in Example 10 the domain of this function is the interval $(-2, 2)$, so we choose the viewing rectangle $[-3, 3]$ by $[-3, 3]$. The graph is shown in Figure 10, and from it we see that the lines $x = -2$ and $x = 2$ are vertical asymptotes.

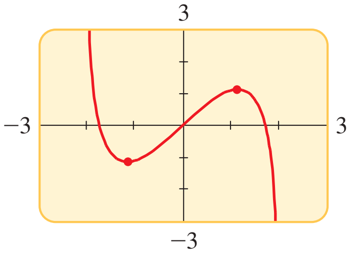


FIGURE 10
$y = x \ln(4 - x^2)$

DISCOVERY PROJECT

Orders of Magnitude

In this project we explore how to compare the sizes of real-world objects using logarithms. For example, how much bigger is an elephant than a flea? How much smaller is a man than a giant redwood? It is difficult to compare objects of such enormously varying sizes. In this project we learn how logarithms can be used to define the concept of "order of magnitude," which provides a simple and meaningful way of comparison. You can find the project at **www.stewartmath.com**.

The function has a local maximum point to the right of $x = 1$ and a local minimum point to the left of $x = -1$. By zooming in and tracing along the graph with the cursor, we find that the local maximum value is approximately 1.13 and this occurs when $x \approx 1.15$. Similarly (or by noticing that the function is odd), we find that the local minimum value is about -1.13, and it occurs when $x \approx -1.15$.

Now Try Exercise 79 ■

4.3 EXERCISES

CONCEPTS

1. log x is the exponent to which the base 10 must be raised to get ________. So we can complete the following table for log x.

x	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	$10^{1/2}$
log x								

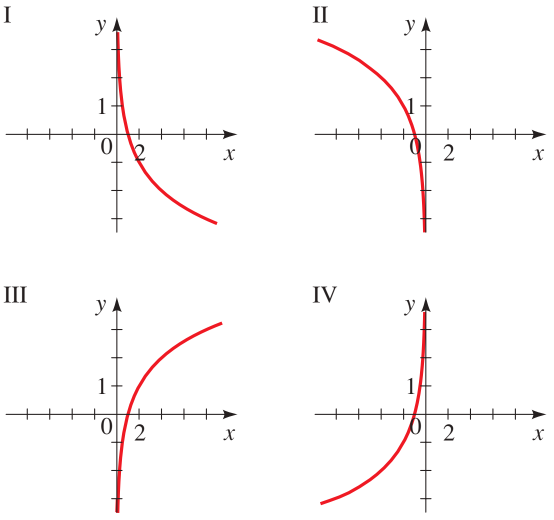
2. The function $f(x) = \log_9 x$ is the logarithm function with base ________. So $f(9) =$ ________, $f(1) =$ ________, $f(\frac{1}{9}) =$ ________, $f(81) =$ ________, and $f(3) =$ ________.

3. (a) $5^3 = 125$, so $\log_{\square} \square = \square$

(b) $\log_5 25 = 2$, so $\square^{\square} = \square$

4. Match the logarithmic function with its graph.

(a) $f(x) = \log_2 x$ **(b)** $f(x) = \log_2(-x)$
(c) $f(x) = -\log_2 x$ **(d)** $f(x) = -\log_2(-x)$



5. The natural logarithmic function $f(x) = \ln x$ has the ____________ asymptote $x =$ _____.

6. The logarithmic function $f(x) = \ln(x - 1)$ has the ____________ asymptote $x =$ _____.

SKILLS

7–8 ■ Logarithmic and Exponential Forms Complete the table by finding the appropriate logarithmic or exponential form of the equation, as in Example 1.

7.

Logarithmic form	Exponential form
$\log_8 8 = 1$	
$\log_8 64 = 2$	
	$8^{2/3} = 4$
	$8^3 = 512$
$\log_8(\frac{1}{8}) = -1$	
	$8^{-2} = \frac{1}{64}$

8.

Logarithmic form	Exponential form
	$4^3 = 64$
$\log_4 2 = \frac{1}{2}$	
	$4^{3/2} = 8$
$\log_4(\frac{1}{16}) = -2$	
$\log_4(\frac{1}{2}) = -\frac{1}{2}$	
	$4^{-5/2} = \frac{1}{32}$

9–16 ■ Exponential Form Express the equation in exponential form.

9. (a) $\log_3 81 = 4$ **(b)** $\log_3 1 = 0$

10. (a) $\log_5(\frac{1}{5}) = -1$ **(b)** $\log_4 64 = 3$

11. (a) $\log_8 2 = \frac{1}{3}$ **(b)** $\log_{10} 0.01 = -2$

12. (a) $\log_5(\frac{1}{125}) = -3$ **(b)** $\log_8 4 = \frac{2}{3}$

13. (a) $\log_3 5 = x$ **(b)** $\log_7(3y) = 2$

14. (a) $\log_6 z = 1$ **(b)** $\log_{10} 3 = 2t$

15. (a) $\ln 5 = 3y$ **(b)** $\ln(t + 1) = -1$

16. (a) $\ln(x + 1) = 2$ **(b)** $\ln(x - 1) = 4$

17–24 ■ Logarithmic Form Express the equation in logarithmic form.

17. (a) $10^4 = 10{,}000$ **(b)** $5^{-2} = \frac{1}{25}$

18. (a) $6^2 = 36$ **(b)** $10^{-1} = \frac{1}{10}$

19. **(a)** $8^{-1} = \frac{1}{8}$ **(b)** $2^{-3} = \frac{1}{8}$

20. **(a)** $4^{-3/2} = 0.125$ **(b)** $7^3 = 343$

21. **(a)** $4^x = 70$ **(b)** $3^5 = w$

22. **(a)** $3^{2x} = 10$ **(b)** $10^{-4x} = 0.1$

23. **(a)** $e^x = 2$ **(b)** $e^3 = y$

24. **(a)** $e^{x+1} = 0.5$ **(b)** $e^{0.5x} = t$

25–34 ■ Evaluating Logarithms Evaluate the expression.

25. **(a)** $\log_2 2$ **(b)** $\log_5 1$ **(c)** $\log_6 6^5$

26. **(a)** $\log_3 3^7$ **(b)** $\log_4 64$ **(c)** $\log_5 125$

27. **(a)** $\log_6 36$ **(b)** $\log_9 81$ **(c)** $\log_7 7^{10}$

28. **(a)** $\log_2 32$ **(b)** $\log_8 8^{17}$ **(c)** $\log_6 1$

29. **(a)** $\log_3\left(\frac{1}{27}\right)$ **(b)** $\log_{10}\sqrt{10}$ **(c)** $\log_5 0.2$

30. **(a)** $\log_5 125$ **(b)** $\log_{49} 7$ **(c)** $\log_9 \sqrt{3}$

31. **(a)** $3^{\log_3 5}$ **(b)** $5^{\log_5 27}$ **(c)** $e^{\ln 10}$

32. **(a)** $e^{\ln\sqrt{3}}$ **(b)** $e^{\ln(1/\pi)}$ **(c)** $10^{\log 13}$

33. **(a)** $\log_8 0.25$ **(b)** $\ln e^4$ **(c)** $\ln(1/e)$

34. **(a)** $\log_4\sqrt{2}$ **(b)** $\log_4\left(\frac{1}{2}\right)$ **(c)** $\log_4 8$

35–44 ■ Logarithmic Equations Use the definition of the logarithmic function to find x.

35. **(a)** $\log_4 x = 3$ **(b)** $\log_{10} 0.01 = x$

36. **(a)** $\log_3 x = -2$ **(b)** $\log_5 125 = x$

37. **(a)** $\ln x = 3$ **(b)** $\ln e^2 = x$

38. **(a)** $\ln x = -1$ **(b)** $\ln(1/e) = x$

39. **(a)** $\log_7\left(\frac{1}{49}\right) = x$ **(b)** $\log_2 x = 5$

40. **(a)** $\log_4 2 = x$ **(b)** $\log_4 x = 2$

41. **(a)** $\log_2\left(\frac{1}{2}\right) = x$ **(b)** $\log_{10} x = -3$

42. **(a)** $\log_x 1000 = 3$ **(b)** $\log_x 25 = 2$

43. **(a)** $\log_x 16 = 4$ **(b)** $\log_x 8 = \frac{3}{2}$

44. **(a)** $\log_x 6 = \frac{1}{2}$ **(b)** $\log_x 3 = \frac{1}{3}$

45–48 ■ Evaluating Logarithms Use a calculator to evaluate the expression, correct to four decimal places.

45. **(a)** $\log 2$ **(b)** $\log 35.2$ **(c)** $\log\left(\frac{2}{3}\right)$

46. **(a)** $\log 50$ **(b)** $\log\sqrt{2}$ **(c)** $\log(3\sqrt{2})$

47. **(a)** $\ln 5$ **(b)** $\ln 25.3$ **(c)** $\ln(1 + \sqrt{3})$

48. **(a)** $\ln 27$ **(b)** $\ln 7.39$ **(c)** $\ln 54.6$

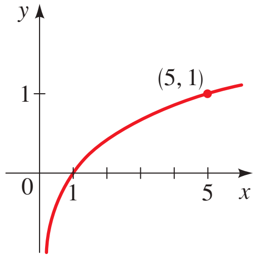
49–52 ■ Graphing Logarithmic Functions Sketch the graph of the function by plotting points.

49. $f(x) = \log_3 x$ **50.** $g(x) = \log_4 x$

51. $f(x) = 2\log x$ **52.** $g(x) = 1 + \log x$

53–56 ■ Finding Logarithmic Functions Find the function of the form $y = \log_a x$ whose graph is given.

53.



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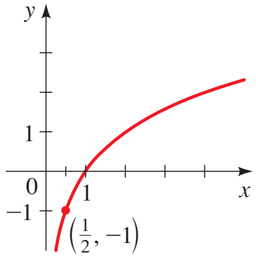

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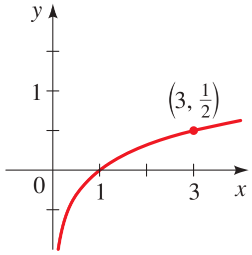

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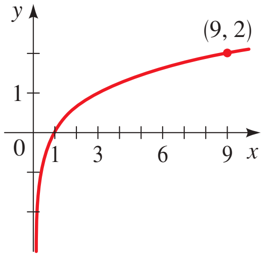
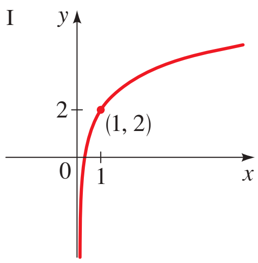
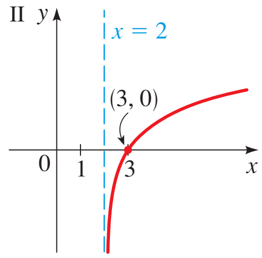


57–58 ■ Graphing Logarithmic Functions Match the logarithmic function with one of the graphs labeled I or II.

57. $f(x) = 2 + \ln x$ **58.** $f(x) = \ln(x - 2)$





59. **Graphing** Draw the graph of $y = 4^x$, then use it to draw the graph of $y = \log_4 x$.

60. **Graphing** Draw the graph of $y = 3^x$, then use it to draw the graph of $y = \log_3 x$.

61–72 ■ Graphing Logarithmic Functions Graph the function, not by plotting points, but by starting from the graphs in Figures 4 and 9. State the domain, range, and asymptote.

61. $g(x) = \log_5(-x)$ **62.** $f(x) = -\log_{10} x$

63. $f(x) = \log_2(x - 4)$ **64.** $g(x) = \ln(x + 2)$

65. $h(x) = \ln(x + 5)$ **66.** $g(x) = \log_6(x - 3)$

67. $y = 2 + \log_3 x$ **68.** $y = 1 - \log_{10} x$

69. $y = \log_3(x - 1) - 2$ **70.** $y = 1 + \ln(-x)$

71. $y = |\ln x|$ **72.** $y = \ln|x|$

73–78 ■ Domain Find the domain of the function.

73. $f(x) = \log_{10}(x + 3)$ **74.** $f(x) = \log_5(8 - 2x)$

75. $g(x) = \log_3(x^2 - 1)$ **76.** $g(x) = \ln(x - x^2)$

77. $h(x) = \ln x + \ln(2 - x)$

78. $h(x) = \sqrt{x - 2} - \log_5(10 - x)$

79–84 ■ Graphing Logarithmic Functions Draw the graph of the function in a suitable viewing rectangle, and use it to find the domain, the asymptotes, and the local maximum and minimum values.

79. $y = \log_{10}(1 - x^2)$

80. $y = \ln(x^2 - x)$

81. $y = x + \ln x$

82. $y = x(\ln x)^2$

83. $y = \dfrac{\ln x}{x}$

84. $y = x \log_{10}(x + 10)$

SKILLS Plus

85–88 ■ Domain of a Composition Find the functions $f \circ g$ and $g \circ f$ and their domains.

85. $f(x) = 2^x, \quad g(x) = x + 1$

86. $f(x) = 3^x, \quad g(x) = x^2 + 1$

87. $f(x) = \log_2 x, \quad g(x) = x - 2$

88. $f(x) = \log x, \quad g(x) = x^2$

89. Rates of Growth Compare the rates of growth of the functions $f(x) = \ln x$ and $g(x) = \sqrt{x}$ by drawing their graphs on a common screen using the viewing rectangle $[-1, 30]$ by $[-1, 6]$.

90. Rates of Growth

(a) By drawing the graphs of the functions

$$f(x) = 1 + \ln(1 + x) \qquad \text{and} \qquad g(x) = \sqrt{x}$$

in a suitable viewing rectangle, show that even when a logarithmic function starts out higher than a root function, it is ultimately overtaken by the root function.

(b) Find, rounded to two decimal places, the solutions of the equation $\sqrt{x} = 1 + \ln(1 + x)$.

91–92 ■ Family of Functions A family of functions is given. **(a)** Draw graphs of the family for $c = 1, 2, 3,$ and 4. **(b)** How are the graphs in part (a) related?

91. $f(x) = \log(cx)$

92. $f(x) = c \log x$

93–94 ■ Inverse Functions A function $f(x)$ is given. **(a)** Find the domain of the function f. **(b)** Find the inverse function of f.

93. $f(x) = \log_2(\log_{10} x)$

94. $f(x) = \ln(\ln(\ln x))$

95. Inverse Functions

(a) Find the inverse of the function $f(x) = \dfrac{2^x}{1 + 2^x}$.

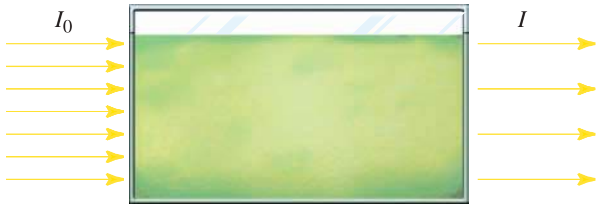
(b) What is the domain of the inverse function?

APPLICATIONS

96. Absorption of Light A spectrophotometer measures the concentration of a sample dissolved in water by shining a light through it and recording the amount of light that emerges. In other words, if we know the amount of light that is absorbed, we can calculate the concentration of the sample. For a certain substance the concentration (in moles per liter, mol/L) is found by using the formula

$$C = -2500 \ln\left(\frac{I}{I_0}\right)$$

where I_0 is the intensity of the incident light and I is the intensity of light that emerges. Find the concentration of the substance if the intensity I is 70% of I_0.



97. Carbon Dating The age of an ancient artifact can be determined by the amount of radioactive carbon-14 remaining in it. If D_0 is the original amount of carbon-14 and D is the amount remaining, then the artifact's age A (in years) is given by

$$A = -8267 \ln\left(\frac{D}{D_0}\right)$$

Find the age of an object if the amount D of carbon-14 that remains in the object is 73% of the original amount D_0.

98. Bacteria Colony A certain strain of bacteria divides every 3 hours. If a colony is started with 50 bacteria, then the time t (in hours) required for the colony to grow to N bacteria is given by

$$t = 3\frac{\log(N/50)}{\log 2}$$

Find the time required for the colony to grow to a million bacteria.

99. Investment The time required to double the amount of an investment at an interest rate r compounded continuously is given by

$$t = \frac{\ln 2}{r}$$

Find the time required to double an investment at 6%, 7%, and 8%.

100. Charging a Battery The rate at which a battery charges is slower the closer the battery is to its maximum charge C_0. The time (in hours) required to charge a fully discharged battery to a charge C is given by

$$t = -k \ln\left(1 - \frac{C}{C_0}\right)$$

where k is a positive constant that depends on the battery. For a certain battery, $k = 0.25$. If this battery is fully discharged, how long will it take to charge to 90% of its maximum charge C_0?

101. Difficulty of a Task The difficulty in "acquiring a target" (such as using your mouse to click on an icon on your computer screen) depends on the distance to the target and the size of the target. According to Fitts's Law, the index of difficulty (ID) is given by

$$\text{ID} = \frac{\log(2A/W)}{\log 2}$$

where W is the width of the target and A is the distance to the center of the target. Compare the difficulty of clicking on an icon that is 5 mm wide to clicking on one that is 10 mm wide. In each case, assume that the mouse is 100 mm from the icon.

DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

102. DISCUSS: The Height of the Graph of a Logarithmic Function Suppose that the graph of $y = 2^x$ is drawn on a coordinate plane where the unit of measurement is an inch.

(a) Show that at a distance 2 ft to the right of the origin the height of the graph is about 265 mi.

(b) If the graph of $y = \log_2 x$ is drawn on the same set of axes, how far to the right of the origin do we have to go before the height of the curve reaches 2 ft?

103. DISCUSS: The Googolplex A **googol** is 10^{100}, and a **googolplex** is 10^{googol}. Find

$$\log(\log(\text{googol})) \qquad \text{and} \qquad \log(\log(\log(\text{googolplex})))$$

104. DISCUSS: Comparing Logarithms Which is larger, $\log_4 17$ or $\log_5 24$? Explain your reasoning.

105. DISCUSS ■ DISCOVER: The Number of Digits in an Integer Compare log 1000 to the number of digits in 1000. Do the same for 10,000. How many digits does any number between 1000 and 10,000 have? Between what two values must the common logarithm of such a number lie? Use your observations to explain why the number of digits in any positive integer x is $[\![\log x]\!] + 1$. (The symbol $[\![n]\!]$ is the greatest integer function defined in Section 2.2.) How many digits does the number 2^{100} have?

4.4 LAWS OF LOGARITHMS

■ Laws of Logarithms ■ Expanding and Combining Logarithmic Expressions
■ Change of Base Formula

In this section we study properties of logarithms. These properties give logarithmic functions a wide range of applications, as we will see in Sections 4.6 and 4.7.

■ Laws of Logarithms

Since logarithms are exponents, the Laws of Exponents give rise to the Laws of Logarithms.

LAWS OF LOGARITHMS

Let a be a positive number, with $a \neq 1$. Let A, B, and C be any real numbers with $A > 0$ and $B > 0$.

Law	Description
1. $\log_a(AB) = \log_a A + \log_a B$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
2. $\log_a\left(\dfrac{A}{B}\right) = \log_a A - \log_a B$	The logarithm of a quotient of numbers is the difference of the logarithms of the numbers.
3. $\log_a(A^C) = C\log_a A$	The logarithm of a power of a number is the exponent times the logarithm of the number.

Proof We make use of the property $\log_a a^x = x$ from Section 4.3.

Law 1 Let $\log_a A = u$ and $\log_a B = v$. When written in exponential form, these equations become

$$a^u = A \quad \text{and} \quad a^v = B$$

Thus

$$\log_a(AB) = \log_a(a^u a^v) = \log_a(a^{u+v})$$
$$= u + v = \log_a A + \log_a B$$

Law 2 Using Law 1, we have

$$\log_a A = \log_a\left[\left(\frac{A}{B}\right)B\right] = \log_a\left(\frac{A}{B}\right) + \log_a B$$

so

$$\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$$

Law 3 Let $\log_a A = u$. Then $a^u = A$, so

$$\log_a(A^C) = \log_a(a^u)^C = \log_a(a^{uC}) = uC = C \log_a A$$

■

EXAMPLE 1 ■ Using the Laws of Logarithms to Evaluate Expressions

Evaluate each expression.

(a) $\log_4 2 + \log_4 32$

(b) $\log_2 80 - \log_2 5$

(c) $-\frac{1}{3}\log 8$

SOLUTION

(a) $\log_4 2 + \log_4 32 = \log_4(2 \cdot 32)$ — Law 1

$= \log_4 64 = 3$ — Because $64 = 4^3$

(b) $\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right)$ — Law 2

$= \log_2 16 = 4$ — Because $16 = 2^4$

(c) $-\frac{1}{3}\log 8 = \log 8^{-1/3}$ — Law 3

$= \log\left(\frac{1}{2}\right)$ — Property of negative exponents

≈ -0.301 — Calculator

Now Try Exercises 9, 11, and 13 ■

Expanding and Combining Logarithmic Expressions

The Laws of Logarithms allow us to write the logarithm of a product or a quotient as the sum or difference of logarithms. This process, called *expanding* a logarithmic expression, is illustrated in the next example.

EXAMPLE 2 ■ Expanding Logarithmic Expressions

Use the Laws of Logarithms to expand each expression.

(a) $\log_2(6x)$ **(b)** $\log_5(x^3y^6)$ **(c)** $\ln\left(\frac{ab}{\sqrt[3]{c}}\right)$

SOLUTION

(a) $\log_2(6x) = \log_2 6 + \log_2 x$ — Law 1

(b) $\log_5(x^3y^6) = \log_5 x^3 + \log_5 y^6$ — Law 1

$= 3\log_5 x + 6\log_5 y$ — Law 3

(c) $\ln\left(\dfrac{ab}{\sqrt[3]{c}}\right) = \ln(ab) - \ln\sqrt[3]{c}$ Law 2

$= \ln a + \ln b - \ln c^{1/3}$ Law 1

$= \ln a + \ln b - \frac{1}{3}\ln c$ Law 3

Now Try Exercises 23, 31, and 37 ■

The Laws of Logarithms also allow us to reverse the process of expanding that was done in Example 2. That is, we can write sums and differences of logarithms as a single logarithm. This process, called *combining* logarithmic expressions, is illustrated in the next example.

EXAMPLE 3 ■ Combining Logarithmic Expressions

Use the Laws of Logarithms to combine each expression into a single logarithm.

(a) $3\log x + \frac{1}{2}\log(x+1)$

(b) $3\ln s + \frac{1}{2}\ln t - 4\ln(t^2+1)$

SOLUTION

(a) $3\log x + \frac{1}{2}\log(x+1) = \log x^3 + \log(x+1)^{1/2}$ Law 3

$= \log(x^3(x+1)^{1/2})$ Law 1

(b) $3\ln s + \frac{1}{2}\ln t - 4\ln(t^2+1) = \ln s^3 + \ln t^{1/2} - \ln(t^2+1)^4$ Law 3

$= \ln(s^3t^{1/2}) - \ln(t^2+1)^4$ Law 1

$= \ln\left(\dfrac{s^3\sqrt{t}}{(t^2+1)^4}\right)$ Law 2

Now Try Exercises 51 and 53 ■

Warning Although the Laws of Logarithms tell us how to compute the logarithm of a product or a quotient, *there is no corresponding rule for the logarithm of a sum or a difference*. For instance,

$$\log_a(x+y) \neq \log_a x + \log_a y$$

In fact, we know that the right side is equal to $\log_a(xy)$. Also, don't improperly simplify quotients or powers of logarithms. For instance,

$$\frac{\log 6}{\log 2} \neq \log\left(\frac{6}{2}\right) \quad \text{and} \quad (\log_2 x)^3 \neq 3\log_2 x$$

Logarithmic functions are used to model a variety of situations involving human behavior. One such behavior is how quickly we forget things we have learned. For example, if you learn algebra at a certain performance level (say, 90% on a test) and then don't use algebra for a while, how much will you retain after a week, a month, or a year? Hermann Ebbinghaus (1850–1909) studied this phenomenon and formulated the law described in the next example.

Forgetting what we've learned depends on how long ago we learned it.

EXAMPLE 4 ■ The Law of Forgetting

If a task is learned at a performance level P_0, then after a time interval t the performance level P satisfies

$$\log P = \log P_0 - c\log(t+1)$$

where c is a constant that depends on the type of task and t is measured in months.

(a) Solve for P.

(b) If your score on a history test is 90, what score would you expect to get on a similar test after two months? After a year? (Assume that $c = 0.2$.)

SOLUTION

(a) We first combine the right-hand side.

$$\log P = \log P_0 - c \log(t + 1) \qquad \text{Given equation}$$

$$\log P = \log P_0 - \log(t + 1)^c \qquad \text{Law 3}$$

$$\log P = \log \frac{P_0}{(t + 1)^c} \qquad \text{Law 2}$$

$$P = \frac{P_0}{(t + 1)^c} \qquad \text{Because log is one-to-one}$$

(b) Here $P_0 = 90$, $c = 0.2$, and t is measured in months.

In 2 months: $t = 2$ and $P = \dfrac{90}{(2 + 1)^{0.2}} \approx 72$

In 1 year: $t = 12$ and $P = \dfrac{90}{(12 + 1)^{0.2}} \approx 54$

Your expected scores after 2 months and after 1 year are 72 and 54, respectively.

Now Try Exercise 73 ■

■ Change of Base Formula

For some purposes we find it useful to change from logarithms in one base to logarithms in another base. Suppose we are given $\log_a x$ and want to find $\log_b x$. Let

$$y = \log_b x$$

We write this in exponential form and take the logarithm, with base a, of each side.

$$b^y = x \qquad \text{Exponential form}$$

$$\log_a(b^y) = \log_a x \qquad \text{Take } \log_a \text{ of each side}$$

$$y \log_a b = \log_a x \qquad \text{Law 3}$$

$$y = \frac{\log_a x}{\log_a b} \qquad \text{Divide by } \log_a b$$

This proves the following formula.

We may write the Change of Base Formula as

$$\log_b x = \left(\frac{1}{\log_a b}\right) \log_a x$$

So $\log_b x$ is just a constant multiple of $\log_a x$; the constant is $\dfrac{1}{\log_a b}$.

CHANGE OF BASE FORMULA

$$\log_b x = \frac{\log_a x}{\log_a b}$$

In particular, if we put $x = a$, then $\log_a a = 1$, and this formula becomes

$$\log_b a = \frac{1}{\log_a b}$$

We can now evaluate a logarithm to *any* base by using the Change of Base Formula to express the logarithm in terms of common logarithms or natural logarithms and then using a calculator.

EXAMPLE 5 ■ Evaluating Logarithms with the Change of Base Formula

Use the Change of Base Formula and common or natural logarithms to evaluate each logarithm, rounded to five decimal places.

(a) $\log_8 5$ **(b)** $\log_9 20$

SOLUTION

(a) We use the Change of Base Formula with $b = 8$ and $a = 10$:

$$\log_8 5 = \frac{\log_{10} 5}{\log_{10} 8} \approx 0.77398$$

(b) We use the Change of Base Formula with $b = 9$ and $a = e$:

$$\log_9 20 = \frac{\ln 20}{\ln 9} \approx 1.36342$$

Now Try Exercises 59 and 61

EXAMPLE 6 ■ Using the Change of Base Formula to Graph a Logarithmic Function

Use a graphing calculator to graph $f(x) = \log_6 x$.

SOLUTION Calculators don't have a key for $\log_6$, so we use the Change of Base Formula to write

$$f(x) = \log_6 x = \frac{\ln x}{\ln 6}$$

Since calculators do have an LN key, we can enter this new form of the function and graph it. The graph is shown in Figure 1.

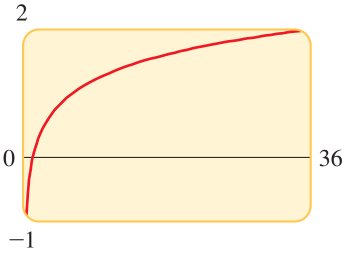


FIGURE 1

$f(x) = \log_6 x = \dfrac{\ln x}{\ln 6}$

Now Try Exercise 67

4.4 EXERCISES

CONCEPTS

1. The logarithm of a product of two numbers is the same as the ________ of the logarithms of these numbers. So $\log_5(25 \cdot 125) =$ ________ + ________.

2. The logarithm of a quotient of two numbers is the same as the ________ of the logarithms of these numbers. So $\log_5(\frac{25}{125}) =$ ________ − ________.

3. The logarithm of a number raised to a power is the same as the ________ times the logarithm of the number. So $\log_5(25^{10}) =$ ________ · ________.

4. We can expand $\log\left(\dfrac{x^2 y}{z}\right)$ to get ________________.

5. We can combine $2\log x + \log y - \log z$ to get ________.

6. (a) Most calculators can find logarithms with base ________ and base ________. To find logarithms with different bases, we use the ________________ Formula. To find $\log_7 12$, we write

$$\log_7 12 = \frac{\log \square}{\log \square} \approx \text{\_\_\_\_\_\_\_\_}$$

(b) Do we get the same answer if we perform the calculation in part (a) using ln in place of log?

7–8 ■ *True or False*?

7. (a) $\log(A + B)$ is the same as $\log A + \log B$.

(b) $\log AB$ is the same as $\log A + \log B$.

8. (a) $\log \dfrac{A}{B}$ is the same as $\log A - \log B$.

(b) $\dfrac{\log A}{\log B}$ is the same as $\log A - \log B$.

SKILLS

9–22 ■ Evaluating Logarithms Use the Laws of Logarithms to evaluate the expression.

9. $\log 50 + \log 200$
10. $\log_6 9 + \log_6 24$
11. $\log_2 60 - \log_2 15$
12. $\log_3 135 - \log_3 45$
13. $\frac{1}{4}\log_3 81$
14. $-\frac{1}{3}\log_3 27$
15. $\log_5 \sqrt{5}$
16. $\log_5 \dfrac{1}{\sqrt{125}}$
17. $\log_2 6 - \log_2 15 + \log_2 20$
18. $\log_3 100 - \log_3 18 - \log_3 50$
19. $\log_4 16^{100}$
20. $\log_2 8^{33}$
21. $\log(\log 10^{10,000})$
22. $\ln(\ln e^{e^{200}})$

23–48 ■ Expanding Logarithmic Expressions Use the Laws of Logarithms to expand the expression.

23. $\log_3 8x$
24. $\log_6 7r$
25. $\log_3 2xy$
26. $\log_5 4st$
27. $\ln a^3$
28. $\log \sqrt{t^5}$
29. $\log_2(xy)^{10}$
30. $\ln \sqrt{ab}$
31. $\log_2(AB^2)$
32. $\log_3(x\sqrt{y})$
33. $\log_3 \dfrac{2x}{y}$
34. $\ln \dfrac{r}{3s}$
35. $\log_5\left(\dfrac{3x^2}{y^3}\right)$
36. $\log_2\left(\dfrac{s^5}{7t^2}\right)$
37. $\log_3 \dfrac{\sqrt{3x^5}}{y}$
38. $\log \dfrac{y^3}{\sqrt{2x}}$
39. $\log\left(\dfrac{x^3y^4}{z^6}\right)$
40. $\log_a\left(\dfrac{x^2}{yz^3}\right)$
41. $\ln \sqrt{x^4 + 2}$
42. $\log \sqrt[3]{x^2 + 4}$
43. $\ln\left(x\sqrt{\dfrac{y}{z}}\right)$
44. $\ln \dfrac{3x^2}{(x+1)^{10}}$
45. $\log \sqrt[4]{x^2 + y^2}$
46. $\log\left(\dfrac{x}{\sqrt[3]{1-x}}\right)$
47. $\log \sqrt{\dfrac{x^2 + 4}{(x^2+1)(x^3-7)^2}}$
48. $\log \sqrt{x\sqrt{y\sqrt{z}}}$

49–58 ■ Combining Logarithmic Expressions Use the Laws of Logarithms to combine the expression.

49. $\log_4 6 + 2\log_4 7$
50. $\frac{1}{2}\log_2 5 - 2\log_2 7$
51. $2\log x - 3\log(x+1)$
52. $3\ln 2 + 2\ln x - \frac{1}{2}\ln(x+4)$
53. $4\log x - \frac{1}{3}\log(x^2+1) + 2\log(x-1)$
54. $\log_5(x^2-1) - \log_5(x-1)$
55. $\ln(a+b) + \ln(a-b) - 2\ln c$
56. $2(\log_5 x + 2\log_5 y - 3\log_5 z)$
57. $\frac{1}{3}\log(x+2)^3 + \frac{1}{2}[\log x^4 - \log(x^2 - x - 6)^2]$
58. $\log_a b + c\log_a d - r\log_a s$

59–66 ■ Change of Base Formula Use the Change of Base Formula and a calculator to evaluate the logarithm, rounded to six decimal places. Use either natural or common logarithms.

59. $\log_2 5$
60. $\log_5 2$
61. $\log_3 16$
62. $\log_6 92$
63. $\log_7 2.61$
64. $\log_6 532$
65. $\log_4 125$
66. $\log_{12} 2.5$

67. **Change of Base Formula** Use the Change of Base Formula to show that

$$\log_3 x = \frac{\ln x}{\ln 3}$$

Then use this fact to draw the graph of the function $f(x) = \log_3 x$.

SKILLS Plus

68. **Families of Functions** Draw graphs of the family of functions $y = \log_a x$ for $a = 2$, e, 5, and 10 on the same screen, using the viewing rectangle $[0, 5]$ by $[-3, 3]$. How are these graphs related?

69. **Change of Base Formula** Use the Change of Base Formula to show that

$$\log e = \frac{1}{\ln 10}$$

70. **Change of Base Formula** Simplify: $(\log_2 5)(\log_5 7)$

71. **A Logarithmic Identity** Show that

$$-\ln(x - \sqrt{x^2 - 1}) = \ln(x + \sqrt{x^2 - 1})$$

APPLICATIONS

72. **Forgetting** Use the Law of Forgetting (Example 4) to estimate a student's score on a biology test two years after he got a score of 80 on a test covering the same material. Assume that $c = 0.3$ and t is measured in months.

73. **Wealth Distribution** Vilfredo Pareto (1848–1923) observed that most of the wealth of a country is owned by a few members of the population. **Pareto's Principle** is

$$\log P = \log c - k\log W$$

where W is the wealth level (how much money a person has) and P is the number of people in the population having that much money.

(a) Solve the equation for P.

(b) Assume that $k = 2.1$ and $c = 8000$, and that W is measured in millions of dollars. Use part (a) to find the number of people who have \$2 million or more. How many people have \$10 million or more?

74. Biodiversity Some biologists model the number of species S in a fixed area A (such as an island) by the species-area relationship

$$\log S = \log c + k \log A$$

where c and k are positive constants that depend on the type of species and habitat.

(a) Solve the equation for S.

(b) Use part (a) to show that if $k = 3$, then doubling the area increases the number of species eightfold.

75. Magnitude of Stars The magnitude M of a star is a measure of how bright a star appears to the human eye. It is defined by

$$M = -2.5 \log\left(\frac{B}{B_0}\right)$$

where B is the actual brightness of the star and B_0 is a constant.

(a) Expand the right-hand side of the equation.

(b) Use part (a) to show that the brighter a star, the less its magnitude.

(c) Betelgeuse is about 100 times brighter than Albiero. Use part (a) to show that Betelgeuse is 5 magnitudes less bright than Albiero.

DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

76. DISCUSS: True or False? Discuss each equation, and determine whether it is true for all possible values of the variables. (Ignore values of the variables for which any term is undefined.)

(a) $\log\left(\frac{x}{y}\right) = \frac{\log x}{\log y}$

(b) $\log_2(x - y) = \log_2 x - \log_2 y$

(c) $\log_5\left(\frac{a}{b^2}\right) = \log_5 a - 2\log_5 b$

(d) $\log 2^z = z \log 2$

(e) $(\log P)(\log Q) = \log P + \log Q$

(f) $\frac{\log a}{\log b} = \log a - \log b$

(g) $(\log_2 7)^x = x \log_2 7$

(h) $\log_a a^a = a$

(i) $\log(x - y) = \frac{\log x}{\log y}$

(j) $-\ln\left(\frac{1}{A}\right) = \ln A$

77. DISCUSS: Find the Error What is wrong with the following argument?

$$\begin{aligned} \log 0.1 &< 2 \log 0.1 \\ &= \log(0.1)^2 \\ &= \log 0.01 \\ \log 0.1 &< \log 0.01 \\ 0.1 &< 0.01 \end{aligned}$$

78. PROVE: Shifting, Shrinking, and Stretching Graphs of Functions Let $f(x) = x^2$. Show that $f(2x) = 4f(x)$, and explain how this shows that shrinking the graph of f horizontally has the same effect as stretching it vertically. Then use the identities $e^{2+x} = e^2e^x$ and $\ln(2x) = \ln 2 + \ln x$ to show that for $g(x) = e^x$ a horizontal shift is the same as a vertical stretch and for $h(x) = \ln x$ a horizontal shrinking is the same as a vertical shift.

4.5 EXPONENTIAL AND LOGARITHMIC EQUATIONS

■ Exponential Equations ■ Logarithmic Equations ■ Compound Interest

In this section we solve equations that involve exponential or logarithmic functions. The techniques that we develop here will be used in the next section for solving applied problems.

■ Exponential Equations

An *exponential equation* is one in which the variable occurs in the exponent. Some exponential equations can be solved by using the fact that exponential functions are one-to-one. This means that

$$a^x = a^y \quad \Rightarrow \quad x = y$$

We use this property in the next example.

EXAMPLE 1 ■ Exponential Equations

Solve the exponential equation.

(a) $5^x = 125$ **(b)** $5^{2x} = 5^{x+1}$

SOLUTION

(a) We first express 125 as a power of 5 and then use the fact that the exponential function $f(x) = 5^x$ is one-to-one.

$$5^x = 125 \quad \text{Given equation}$$
$$5^x = 5^3 \quad \text{Because } 125 = 5^3$$
$$x = 3 \quad \text{One-to-one property}$$

The solution is $x = 3$.

(b) We first use the fact that the function $f(x) = 5^x$ is one-to-one.

$$5^{2x} = 5^{x+1} \quad \text{Given equation}$$
$$2x = x + 1 \quad \text{One-to-one property}$$
$$x = 1 \quad \text{Solve for } x$$

The solution is $x = 1$.

Now Try Exercises 3 and 7 ■

Law 3: $\log_a A^C = C \log_a A$

The equations in Example 1 were solved by comparing exponents. This method is not suitable for solving an equation like $5^x = 160$ because 160 is not easily expressed as a power of the base 5. To solve such equations, we take the logarithm of each side and use Law 3 of logarithms to "bring down the exponent." The following guidelines describe the process.

GUIDELINES FOR SOLVING EXPONENTIAL EQUATIONS

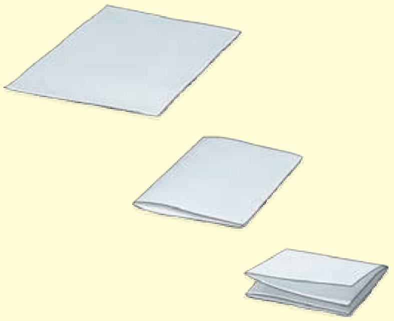
1. Isolate the exponential expression on one side of the equation.
2. Take the logarithm of each side, then use the Laws of Logarithms to "bring down the exponent."
3. Solve for the variable.

EXAMPLE 2 ■ Solving an Exponential Equation

Consider the exponential equation $3^{x+2} = 7$.

(a) Find the exact solution of the equation expressed in terms of logarithms.

(b) Use a calculator to find an approximation to the solution rounded to six decimal places.



DISCOVERY PROJECT

Super Origami

Origami is the traditional Japanese art of folding paper to create illustrations. In this project we explore some thought experiments about folding paper. Suppose that you fold a sheet of paper in half, then fold it in half again, and continue to fold the paper in half. How many folds are needed to obtain a mile-high stack of paper? To answer this question, we need to solve an exponential equation. In this project we use logarithms to answer this and other thought questions about folding paper. You can find the project at **www.stewartmath.com**.

We could have used natural logarithms instead of common logarithms. In fact, using the same steps, we get

$$x = \frac{\ln 7}{\ln 3} - 2 \approx -0.228756$$

CHECK YOUR ANSWER

Substituting $x = -0.228756$ into the original equation and using a calculator, we get

$$3^{(-0.228756)+2} \approx 7 \quad ✓$$

SOLUTION

(a) We take the common logarithm of each side and use Law 3.

$$\begin{aligned} 3^{x+2} &= 7 && \text{Given equation} \\ \log(3^{x+2}) &= \log 7 && \text{Take log of each side} \\ (x+2)\log 3 &= \log 7 && \text{Law 3 (bring down exponent)} \\ x + 2 &= \frac{\log 7}{\log 3} && \text{Divide by log 3} \\ x &= \frac{\log 7}{\log 3} - 2 && \text{Subtract 2} \end{aligned}$$

The exact solution is $x = \dfrac{\log 7}{\log 3} - 2$.

(b) Using a calculator, we find the decimal approximation $x \approx -0.228756$.

Now Try Exercise 15

EXAMPLE 3 ■ Solving an Exponential Equation

Solve the equation $8e^{2x} = 20$.

SOLUTION We first divide by 8 to isolate the exponential term on one side of the equation.

$$\begin{aligned} 8e^{2x} &= 20 && \text{Given equation} \\ e^{2x} &= \tfrac{20}{8} && \text{Divide by 8} \\ \ln e^{2x} &= \ln 2.5 && \text{Take ln of each side} \\ 2x &= \ln 2.5 && \text{Property of ln} \\ x &= \frac{\ln 2.5}{2} && \text{Divide by 2 (exact solution)} \\ &\approx 0.458 && \text{Calculator (approximate solution)} \end{aligned}$$

Now Try Exercise 17

CHECK YOUR ANSWER

Substituting $x = 0.458$ into the original equation and using a calculator, we get

$$8e^{2(0.458)} \approx 20 \quad ✓$$

EXAMPLE 4 ■ Solving an Exponential Equation Algebraically and Graphically

Solve the equation $e^{3-2x} = 4$ algebraically and graphically.

SOLUTION 1: Algebraic

Since the base of the exponential term is e, we use natural logarithms to solve this equation.

$$\begin{aligned} e^{3-2x} &= 4 && \text{Given equation} \\ \ln(e^{3-2x}) &= \ln 4 && \text{Take ln of each side} \\ 3 - 2x &= \ln 4 && \text{Property of ln} \\ -2x &= -3 + \ln 4 && \text{Subtract 3} \\ x &= \tfrac{1}{2}(3 - \ln 4) \approx 0.807 && \text{Multiply by } -\tfrac{1}{2} \end{aligned}$$

You should check that this answer satisfies the original equation.

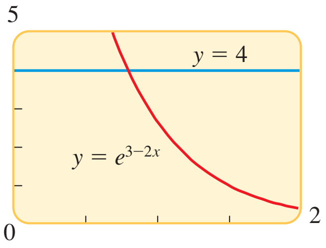


FIGURE 1

SOLUTION 2: Graphical

We graph the equations $y = e^{3-2x}$ and $y = 4$ in the same viewing rectangle as in Figure 1. The solutions occur where the graphs intersect. Zooming in on the point of intersection of the two graphs, we see that $x \approx 0.81$.

Now Try Exercise 21 ■

EXAMPLE 5 ■ An Exponential Equation of Quadratic Type

Solve the equation $e^{2x} - e^x - 6 = 0$.

SOLUTION To isolate the exponential term, we factor.

$$e^{2x} - e^x - 6 = 0 \qquad \text{Given equation}$$

$$(e^x)^2 - e^x - 6 = 0 \qquad \text{Law of Exponents}$$

$$(e^x - 3)(e^x + 2) = 0 \qquad \text{Factor (a quadratic in } e^x\text{)}$$

$$e^x - 3 = 0 \quad \text{or} \quad e^x + 2 = 0 \qquad \text{Zero-Product Property}$$

$$e^x = 3 \qquad\qquad e^x = -2$$

If we let $w = e^x$, we get the quadratic equation

$$w^2 - w - 6 = 0$$

which factors as

$$(w - 3)(w + 2) = 0$$

The equation $e^x = 3$ leads to $x = \ln 3$. But the equation $e^x = -2$ has no solution because $e^x > 0$ for all x. Thus $x = \ln 3 \approx 1.0986$ is the only solution. You should check that this answer satisfies the original equation.

Now Try Exercise 39 ■

EXAMPLE 6 ■ An Equation Involving Exponential Functions

Solve the equation $3xe^x + x^2e^x = 0$.

SOLUTION First we factor the left side of the equation.

$$3xe^x + x^2e^x = 0 \qquad \text{Given equation}$$

$$x(3 + x)e^x = 0 \qquad \text{Factor out common factors}$$

$$x(3 + x) = 0 \qquad \text{Divide by } e^x \text{ (because } e^x \neq 0\text{)}$$

$$x = 0 \quad \text{or} \quad 3 + x = 0 \qquad \text{Zero-Product Property}$$

Thus the solutions are $x = 0$ and $x = -3$.

CHECK YOUR ANSWER

$x = 0$:

$3(0)e^0 + 0^2e^0 = 0$ ✓

$x = -3$:

$3(-3)e^{-3} + (-3)^2e^{-3}$

$= -9e^{-3} + 9e^{-3} = 0$ ✓

Now Try Exercise 45 ■

■ Logarithmic Equations

A *logarithmic equation* is one in which a logarithm of the variable occurs. Some logarithmic equations can be solved by using the fact that logarithmic functions are one-to-one. This means that

$$\log_a x = \log_a y \quad \Rightarrow \quad x = y$$

We use this property in the next example.

EXAMPLE 7 ■ Solving a Logarithmic Equation

Solve the equation $\log(x^2 + 1) = \log(x - 2) + \log(x + 3)$.

SOLUTION First we combine the logarithms on the right-hand side, and then we use the one-to-one property of logarithms.

$\log_5(x^2 + 1) = \log_5(x - 2) + \log_5(x + 3)$	Given equation
$\log_5(x^2 + 1) = \log_5[(x - 2)(x + 3)]$	Law 1: $\log_a AB = \log_a A + \log_a B$
$\log_5(x^2 + 1) = \log_5(x^2 + x - 6)$	Expand
$x^2 + 1 = x^2 + x - 6$	log is one-to-one (or raise 5 to each side)
$x = 7$	Solve for x

The solution is $x = 7$. (You can check that $x = 7$ satisfies the original equation.)

Now Try Exercise 49

The method of Example 7 is not suitable for solving an equation like $\log_5 x = 13$ because the right-hand side is not expressed as a logarithm (base 5). To solve such equations, we use the following guidelines.

GUIDELINES FOR SOLVING LOGARITHMIC EQUATIONS

1. Isolate the logarithmic term on one side of the equation; you might first need to combine the logarithmic terms.
2. Write the equation in exponential form (or raise the base to each side of the equation).
3. Solve for the variable.

EXAMPLE 8 ■ Solving Logarithmic Equations

Solve each equation for x.

(a) $\ln x = 8$

(b) $\log_2(25 - x) = 3$

SOLUTION

(a)

$\ln x = 8$	Given equation
$x = e^8$	Exponential form

Therefore $x = e^8 \approx 2981$.

We can also solve this problem another way.

$\ln x = 8$	Given equation
$e^{\ln x} = e^8$	Raise e to each side
$x = e^8$	Property of ln

(b) The first step is to rewrite the equation in exponential form.

$\log_2(25 - x) = 3$	Given equation
$25 - x = 2^3$	Exponential form (or raise 2 to each side)
$25 - x = 8$	
$x = 25 - 8 = 17$	

CHECK YOUR ANSWER

If $x = 17$, we get

$\log_2(25 - 17) = \log_2 8 = 3$ ✓

Now Try Exercises 55 and 59

EXAMPLE 9 ■ Solving a Logarithmic Equation

Solve the equation $4 + 3\log(2x) = 16$.

SOLUTION We first isolate the logarithmic term. This allows us to write the equation in exponential form.

$$\begin{aligned} 4 + 3\log(2x) &= 16 && \text{Given equation} \\ 3\log(2x) &= 12 && \text{Subtract 4} \\ \log(2x) &= 4 && \text{Divide by 3} \\ 2x &= 10^4 && \text{Exponential form (or raise 10 to each side)} \\ x &= 5000 && \text{Divide by 2} \end{aligned}$$

CHECK YOUR ANSWER

If $x = 5000$, we get

$$\begin{aligned} 4 + 3\log 2(5000) &= 4 + 3\log 10{,}000 \\ &= 4 + 3(4) \\ &= 16 \quad ✓ \end{aligned}$$

Now Try Exercise 61 ■

EXAMPLE 10 ■ Solving a Logarithmic Equation Algebraically and Graphically

Solve the equation $\log(x + 2) + \log(x - 1) = 1$ algebraically and graphically.

SOLUTION 1: Algebraic

We first combine the logarithmic terms, using the Laws of Logarithms.

$$\begin{aligned} \log[(x + 2)(x - 1)] &= 1 && \text{Law 1} \\ (x + 2)(x - 1) &= 10 && \text{Exponential form (or raise 10 to each side)} \\ x^2 + x - 2 &= 10 && \text{Expand left side} \\ x^2 + x - 12 &= 0 && \text{Subtract 10} \\ (x + 4)(x - 3) &= 0 && \text{Factor} \\ x = -4 \quad \text{or} \quad x &= 3 \end{aligned}$$

We check these potential solutions in the original equation and find that $x = -4$ is not a solution (because logarithms of negative numbers are undefined), but $x = 3$ is a solution. (See *Check Your Answers*.)

CHECK YOUR ANSWERS

$x = -4$:

$$\begin{aligned} &\log(-4 + 2) + \log(-4 - 1) \\ &= \log(-2) + \log(-5) \\ &\qquad \text{undefined} \quad ✗ \end{aligned}$$

$x = 3$:

$$\begin{aligned} &\log(3 + 2) + \log(3 - 1) \\ &= \log 5 + \log 2 = \log(5 \cdot 2) \\ &= \log 10 = 1 \quad ✓ \end{aligned}$$

SOLUTION 2: Graphical

We first move all terms to one side of the equation:

$$\log(x + 2) + \log(x - 1) - 1 = 0$$

Then we graph

$$y = \log(x + 2) + \log(x - 1) - 1$$

as in Figure 2. The solutions are the x-intercepts of the graph. Thus the only solution is $x \approx 3$.

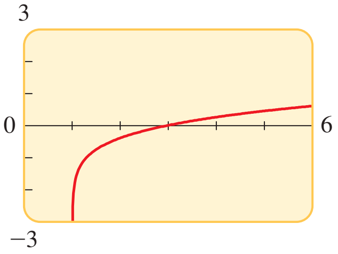


FIGURE 2

Now Try Exercise 63 ■

EXAMPLE 11 ■ Solving a Logarithmic Equation Graphically

In Example 11 it's not possible to isolate x algebraically, so we must solve the equation graphically.

Solve the equation $x^2 = 2\ln(x + 2)$.

SOLUTION We first move all terms to one side of the equation.

$$x^2 - 2\ln(x + 2) = 0$$

Then we graph

$$y = x^2 - 2\ln(x + 2)$$

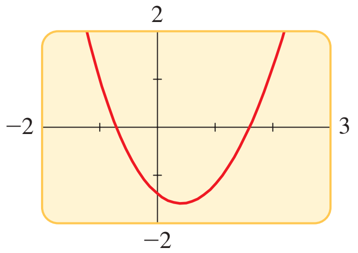


FIGURE 3

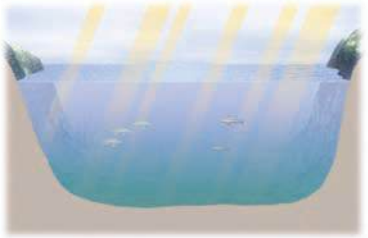
as in Figure 3. The solutions are the x-intercepts of the graph. Zooming in on the x-intercepts, we see that there are two solutions:

$$x \approx -0.71 \qquad \text{and} \qquad x \approx 1.60$$

Now Try Exercise 69 ■

Logarithmic equations are used in determining the amount of light that reaches various depths in a lake. (This information helps biologists to determine the types of life a lake can support.) As light passes through water (or other transparent materials such as glass or plastic), some of the light is absorbed. It's easy to see that the murkier the water, the more light is absorbed. The exact relationship between light absorption and the distance light travels in a material is described in the next example.

EXAMPLE 12 ■ Transparency of a Lake



The intensity of light in a lake diminishes with depth.

If I_0 and I denote the intensity of light before and after going through a material and x is the distance (in feet) the light travels in the material, then according to the **Beer-Lambert Law**,

$$-\frac{1}{k}\ln\left(\frac{I}{I_0}\right) = x$$

where k is a constant depending on the type of material.

(a) Solve the equation for I.

(b) For a certain lake $k = 0.025$, and the light intensity is $I_0 = 14$ lumens (lm). Find the light intensity at a depth of 20 ft.

SOLUTION

(a) We first isolate the logarithmic term.

$$-\frac{1}{k}\ln\left(\frac{I}{I_0}\right) = x \qquad \text{Given equation}$$

$$\ln\left(\frac{I}{I_0}\right) = -kx \qquad \text{Multiply by } -k$$

$$\frac{I}{I_0} = e^{-kx} \qquad \text{Exponential form}$$

$$I = I_0 e^{-kx} \qquad \text{Multiply by } I_0$$

(b) We find I using the formula from part (a).

$$I = I_0 e^{-kx} \qquad \text{From part (a)}$$

$$= 14e^{(-0.025)(20)} \qquad I_0 = 14,\ k = 0.025,\ x = 20$$

$$\approx 8.49 \qquad \text{Calculator}$$

The light intensity at a depth of 20 ft is about 8.5 lm.

Now Try Exercise 99 ■

■ Compound Interest

Recall the formulas for interest that we found in Section 4.1. If a principal P is invested at an interest rate r for a period of t years, then the amount A of the investment is given by

$$A = P(1 + r) \qquad \text{Simple interest (for one year)}$$

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{Interest compounded } n \text{ times per year}$$

$$A(t) = Pe^{rt} \qquad \text{Interest compounded continuously}$$

Radiocarbon Dating is a method that archeologists use to determine the age of ancient objects. The carbon dioxide in the atmosphere always contains a fixed fraction of radioactive carbon, carbon-14 (^{14}C), with a half-life of about 5730 years. Plants absorb carbon dioxide from the atmosphere, which then makes its way to animals through the food chain. Thus, all living creatures contain the same fixed proportions of ^{14}C to nonradioactive ^{12}C as the atmosphere.

After an organism dies, it stops assimilating ^{14}C, and the amount of ^{14}C in it begins to decay exponentially. We can then determine the time that has elapsed since the death of the organism by measuring the amount of ^{14}C left in it.

For example, if a donkey bone contains 73% as much ^{14}C as a living donkey and it died t years ago, then by the formula for radioactive decay (Section 4.6),

$$0.73 = (1.00)e^{-(t \ln 2)/5730}$$

We solve this exponential equation to find $t \approx 2600$, so the bone is about 2600 years old.

We can use logarithms to determine the time it takes for the principal to increase to a given amount.

EXAMPLE 13 ■ Finding the Term for an Investment to Double

A sum of \$5000 is invested at an interest rate of 5% per year. Find the time required for the money to double if the interest is compounded according to the following methods.

(a) Semiannually **(b)** Continuously

SOLUTION

(a) We use the formula for compound interest with $P = \$5000$, $A(t) = \$10,000$, $r = 0.05$, and $n = 2$, and solve the resulting exponential equation for t.

$$
\begin{aligned}
5000\left(1 + \frac{0.05}{2}\right)^{2t} &= 10,000 && P\left(1+\frac{r}{n}\right)^{nt} = A \\
(1.025)^{2t} &= 2 && \text{Divide by 5000} \\
\log 1.025^{2t} &= \log 2 && \text{Take log of each side} \\
2t \log 1.025 &= \log 2 && \text{Law 3 (bring down the exponent)} \\
t &= \frac{\log 2}{2 \log 1.025} && \text{Divide by 2 log 1.025} \\
t &\approx 14.04 && \text{Calculator}
\end{aligned}
$$

The money will double in 14.04 years.

(b) We use the formula for continuously compounded interest with $P = \$5000$, $A(t) = \$10,000$, and $r = 0.05$ and solve the resulting exponential equation for t.

$$
\begin{aligned}
5000e^{0.05t} &= 10,000 && Pe^{rt} = A \\
e^{0.05t} &= 2 && \text{Divide by 5000} \\
\ln e^{0.05t} &= \ln 2 && \text{Take ln of each side} \\
0.05t &= \ln 2 && \text{Property of ln} \\
t &= \frac{\ln 2}{0.05} && \text{Divide by 0.05} \\
t &\approx 13.86 && \text{Calculator}
\end{aligned}
$$

The money will double in 13.86 years.

Now Try Exercise 89 ■

EXAMPLE 14 ■ Time Required to Grow an Investment

A sum of \$1000 is invested at an interest rate of 4% per year. Find the time required for the amount to grow to \$4000 if interest is compounded continuously.

SOLUTION We use the formula for continuously compounded interest with $P = \$1000$, $A(t) = \$4000$, and $r = 0.04$ and solve the resulting exponential equation for t.

$$
\begin{aligned}
1000e^{0.04t} &= 4000 && Pe^{rt} = A \\
e^{0.04t} &= 4 && \text{Divide by 1000} \\
0.04t &= \ln 4 && \text{Take ln of each side} \\
t &= \frac{\ln 4}{0.04} && \text{Divide by 0.04} \\
t &\approx 34.66 && \text{Calculator}
\end{aligned}
$$

The amount will be \$4000 in about 34 years and 8 months.

Now Try Exercise 91 ■

4.5 EXERCISES

CONCEPTS

1. Let's solve the exponential equation $2e^x = 50$.

(a) First, we isolate e^x to get the equivalent equation ________.

(b) Next, we take ln of each side to get the equivalent equation ________________.

(c) Now we use a calculator to find $x \approx$ ________.

2. Let's solve the logarithmic equation

$$\log 3 + \log(x - 2) = \log x$$

(a) First, we combine the logarithms on the LHS to get the equivalent equation ________________.

(b) Next, we use the fact that log is one-to-one to get the equivalent equation ________________.

(c) Now we find $x =$ ________.

SKILLS

3–10 ■ Exponential Equations Find the solution of the exponential equation, as in Example 1.

3. $5^{x-1} = 125$ **4.** $e^{x^2} = e^9$

5. $5^{2x-3} = 1$ **6.** $10^{2x-3} = \frac{1}{10}$

7. $7^{2x-3} = 7^{6+5x}$ **8.** $e^{1-2x} = e^{3x-5}$

9. $6^{x^2-1} = 6^{1-x^2}$ **10.** $10^{2x^2-3} = 10^{9-x^2}$

11–38 ■ Exponential Equations **(a)** Find the exact solution of the exponential equation in terms of logarithms. **(b)** Use a calculator to find an approximation to the solution rounded to six decimal places.

11. $10^x = 25$ **12.** $10^{-x} = 4$

13. $e^{-5x} = 10$ **14.** $e^{0.4x} = 8$

15. $2^{1-x} = 3$ **16.** $3^{2x-1} = 5$

17. $3e^x = 10$ **18.** $2e^{12x} = 17$

19. $300(1.025)^{12t} = 1000$ **20.** $10(1.375)^{10t} = 50$

21. $e^{1-4x} = 2$ **22.** $e^{3-5x} = 16$

23. $2^{5-7x} = 15$ **24.** $2^{3x} = 34$

25. $3^{x/14} = 0.1$ **26.** $5^{-x/100} = 2$

27. $4(1 + 10^{5x}) = 9$ **28.** $2(5 + 3^{x+1}) = 100$

29. $8 + e^{1-4x} = 20$ **30.** $1 + e^{4x+1} = 20$

31. $4^x + 2^{1+2x} = 50$ **32.** $125^x + 5^{3x+1} = 200$

33. $5^x = 4^{x+1}$ **34.** $10^{1-x} = 6^x$

35. $2^{3x+1} = 3^{x-2}$ **36.** $7^{x/2} = 5^{1-x}$

37. $\dfrac{50}{1 + e^{-x}} = 4$ **38.** $\dfrac{10}{1 + e^{-x}} = 2$

39–44 ■ Exponential Equations of Quadratic Type Solve the equation.

39. $e^{2x} - 3e^x + 2 = 0$ **40.** $e^{2x} - e^x - 6 = 0$

41. $e^{4x} + 4e^{2x} - 21 = 0$ **42.** $3^{4x} - 3^{2x} - 6 = 0$

43. $2^x - 10(2^{-x}) + 3 = 0$ **44.** $e^x + 15e^{-x} - 8 = 0$

45–48 ■ Equations Involving Exponential Functions Solve the equation.

45. $x^2 2^x - 2^x = 0$ **46.** $x^2 10^x - x10^x = 2(10^x)$

47. $4x^3e^{-3x} - 3x^4e^{-3x} = 0$ **48.** $x^2e^x + xe^x - e^x = 0$

49–54 ■ Logarithmic Equations Solve the logarithmic equation for x, as in Example 7.

49. $\log x + \log(x - 1) = \log(4x)$

50. $\log_5 x + \log_5(x + 1) = \log_5 20$

51. $2 \log x = \log 2 + \log(3x - 4)$

52. $\ln\left(x - \frac{1}{2}\right) + \ln 2 = 2 \ln x$

53. $\log_2 3 + \log_2 x = \log_2 5 + \log_2(x - 2)$

54. $\log_4(x + 2) + \log_4 3 = \log_4 5 + \log_4(2x - 3)$

55–68 ■ Logarithmic Equations Solve the logarithmic equation for x.

55. $\ln x = 10$ **56.** $\ln(2 + x) = 1$

57. $\log x = -2$ **58.** $\log(x - 4) = 3$

59. $\log(3x + 5) = 2$ **60.** $\log_3(2 - x) = 3$

61. $4 - \log(3 - x) = 3$

62. $\log_2(x^2 - x - 2) = 2$

63. $\log_2 x + \log_2(x - 3) = 2$

64. $\log x + \log(x - 3) = 1$

65. $\log_9(x - 5) + \log_9(x + 3) = 1$

66. $\ln(x - 1) + \ln(x + 2) = 1$

67. $\log_5(x + 1) - \log_5(x - 1) = 2$

68. $\log_3(x + 15) - \log_3(x - 1) = 2$

69–76 ■ Solving Equations Graphically Use a graphing device to find all solutions of the equation, rounded to two decimal places.

69. $\ln x = 3 - x$ **70.** $\log x = x^2 - 2$

71. $x^3 - x = \log(x + 1)$ **72.** $x = \ln(4 - x^2)$

73. $e^x = -x$ **74.** $2^{-x} = x - 1$

75. $4^{-x} = \sqrt{x}$ **76.** $e^{x^2} - 2 = x^3 - x$

77–78 ■ More Exponential and Logarithmic Equations Solve the equation for x.

77. $2^{2/\log_5 x} = \frac{1}{16}$ **78.** $\log_2(\log_3 x) = 4$

SKILLS Plus

79–82 ■ Solving Inequalities Solve the inequality.

79. $\log(x - 2) + \log(9 - x) < 1$

80. $3 \le \log_2 x \le 4$

81. $2 < 10^x < 5$

82. $x^2e^x - 2e^x < 0$

83–86 ■ Inverse Functions Find the inverse function of f.

83. $f(x) = 2^{2x}$

84. $f(x) = 3^{x+1}$

85. $f(x) = \log_2(x - 1)$

86. $f(x) = \log 3x$

87–88 ■ Special Logarithmic Equations Find the value(s) of x for which the equation is true.

87. $\log(x + 3) = \log x + \log 3$

88. $(\log x)^3 = 3 \log x$

APPLICATIONS

89. Compound Interest A man invests \$5000 in an account that pays 8.5% interest per year, compounded quarterly.

(a) Find the amount after 3 years.

(b) How long will it take for the investment to double?

90. Compound Interest A woman invests \$6500 in an account that pays 6% interest per year, compounded continuously.

(a) What is the amount after 2 years?

(b) How long will it take for the amount to be \$8000?

91. Compound Interest Find the time required for an investment of \$5000 to grow to \$8000 at an interest rate of 7.5% per year, compounded quarterly.

92. Compound Interest Nancy wants to invest \$4000 in saving certificates that bear an interest rate of 9.75% per year, compounded semiannually. How long a time period should she choose to save an amount of \$5000?

93. Doubling an Investment How long will it take for an investment of \$1000 to double in value if the interest rate is 8.5% per year, compounded continuously?

94. Interest Rate A sum of \$1000 was invested for 4 years, and the interest was compounded semiannually. If this sum amounted to \$1435.77 in the given time, what was the interest rate?

95. Radioactive Decay A 15-g sample of radioactive iodine decays in such a way that the mass remaining after t days is given by $m(t) = 15e^{-0.087t}$, where $m(t)$ is measured in grams. After how many days are there only 5 g remaining?

96. Sky Diving The velocity of a sky diver t seconds after jumping is given by $v(t) = 80(1 - e^{-0.2t})$. After how many seconds is the velocity 70 ft/s?

97. Fish Population A small lake is stocked with a certain species of fish. The fish population is modeled by the function

$$P = \frac{10}{1 + 4e^{-0.8t}}$$

where P is the number of fish in thousands and t is measured in years since the lake was stocked.

(a) Find the fish population after 3 years.

(b) After how many years will the fish population reach 5000 fish?

98. Transparency of a Lake Environmental scientists measure the intensity of light at various depths in a lake to find the "transparency" of the water. Certain levels of transparency are required for the biodiversity of the submerged macrophyte population. In a certain lake the intensity of light at depth x is given by

$$I = 10e^{-0.008x}$$

where I is measured in lumens and x in feet.

(a) Find the intensity I at a depth of 30 ft.

(b) At what depth has the light intensity dropped to $I = 5$?

99. Atmospheric Pressure Atmospheric pressure P (in kilopascals, kPa) at altitude h (in kilometers, km) is governed by the formula

$$\ln\left(\frac{P}{P_0}\right) = -\frac{h}{k}$$

where $k = 7$ and $P_0 = 100$ kPa are constants.

(a) Solve the equation for P.

(b) Use part (a) to find the pressure P at an altitude of 4 km.

100. Cooling an Engine Suppose you're driving your car on a cold winter day (20°F outside) and the engine overheats (at about 220°F). When you park, the engine begins to cool down. The temperature T of the engine t minutes after you park satisfies the equation

$$\ln\left(\frac{T - 20}{200}\right) = -0.11t$$

(a) Solve the equation for T.

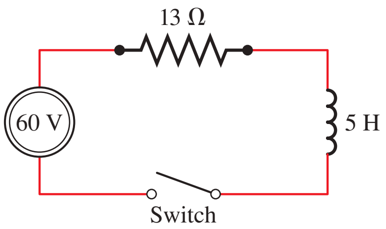
(b) Use part (a) to find the temperature of the engine after 20 min ($t = 20$).

101. Electric Circuits An electric circuit contains a battery that produces a voltage of 60 volts (V), a resistor with a resistance of 13 ohms (Ω), and an inductor with an inductance of 5 henrys (H), as shown in the figure on the following page. Using calculus, it can be shown that the current

$I = I(t)$ (in amperes, A) t seconds after the switch is closed is $I = \frac{60}{13}(1 - e^{-13t/5})$.

(a) Use this equation to express the time t as a function of the current I.

(b) After how many seconds is the current 2 A?



102. Learning Curve A *learning curve* is a graph of a function $P(t)$ that measures the performance of someone learning a skill as a function of the training time t. At first, the rate of learning is rapid. Then, as performance increases and approaches a maximal value M, the rate of learning decreases. It has been found that the function

$$P(t) = M - Ce^{-kt}$$

where k and C are positive constants and $C < M$ is a reasonable model for learning.

(a) Express the learning time t as a function of the performance level P.

(b) For a pole-vaulter in training, the learning curve is given by

$$P(t) = 20 - 14e^{-0.024t}$$

where $P(t)$ is the height he is able to pole-vault after t months. After how many months of training is he able to vault 12 ft?

(c) Draw a graph of the learning curve in part (b).

DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

103. DISCUSS: Estimating a Solution Without actually solving the equation, find two whole numbers between which the solution of $9^x = 20$ must lie. Do the same for $9^x = 100$. Explain how you reached your conclusions.

104. DISCUSS ■ DISCOVER: A Surprising Equation Take logarithms to show that the equation

$$x^{1/\log x} = 5$$

has no solution. For what values of k does the equation

$$x^{1/\log x} = k$$

have a solution? What does this tell us about the graph of the function $f(x) = x^{1/\log x}$? Confirm your answer using a graphing device.

105. DISCUSS: Disguised Equations Each of these equations can be transformed into an equation of linear or quadratic type by applying the hint. Solve each equation.

(a) $(x - 1)^{\log(x-1)} = 100(x - 1)$
[*Hint:* Take log of each side.]

(b) $\log_2 x + \log_4 x + \log_8 x = 11$
[*Hint:* Change all logs to base 2.]

(c) $4^x - 2^{x+1} = 3$
[*Hint:* Write as a quadratic in 2^x.]

4.6 MODELING WITH EXPONENTIAL FUNCTIONS

■ Exponential Growth (Doubling Time) ■ Exponential Growth (Relative Growth Rate) ■ Radioactive Decay ■ Newton's Law of Cooling

Many processes that occur in nature, such as population growth, radioactive decay, heat diffusion, and numerous others, can be modeled by using exponential functions. In this section we study exponential models.

■ Exponential Growth (Doubling Time)

Suppose we start with a single bacterium, which divides every hour. After one hour we have 2 bacteria, after two hours we have 2^2 or 4 bacteria, after three hours we have 2^3

or 8 bacteria, and so on (see Figure 1). We see that we can model the bacteria population after t hours by $f(t) = 2^t$.

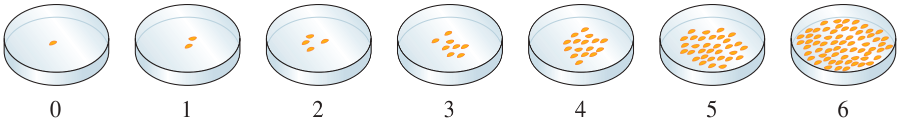


FIGURE 1 Bacteria population

If we start with 10 of these bacteria, then the population is modeled by $f(t) = 10 \cdot 2^t$. A slower-growing strain of bacteria doubles every 3 hours; in this case the population is modeled by $f(t) = 10 \cdot 2^{t/3}$. In general, we have the following.

EXPONENTIAL GROWTH (DOUBLING TIME)

If the initial size of a population is n_0 and the doubling time is a, then the size of the population at time t is

$$n(t) = n_0 2^{t/a}$$

where a and t are measured in the same time units (minutes, hours, days, years, and so on).

EXAMPLE 1 ■ Bacteria Population

Under ideal conditions a certain bacteria population doubles every three hours. Initially, there are 1000 bacteria in a colony.

(a) Find a model for the bacteria population after t hours.

(b) How many bacteria are in the colony after 15 hours?

(c) After how many hours will the bacteria count reach 100,000?

SOLUTION

(a) The population at time t is modeled by

$$n(t) = 1000 \cdot 2^{t/3}$$

where t is measured in hours.

(b) After 15 hours the number of bacteria is

$$n(15) = 1000 \cdot 2^{15/3} = 32{,}000$$

(c) We set $n(t) = 100{,}000$ in the model that we found in part (a) and solve the resulting exponential equation for t.

$$100{,}000 = 1000 \cdot 2^{t/3} \qquad n(t) = 1000 \cdot 2^{t/3}$$

$$100 = 2^{t/3} \qquad \text{Divide by 1000}$$

$$\log 100 = \log 2^{t/3} \qquad \text{Take log of each side}$$

$$2 = \frac{t}{3} \log 2 \qquad \text{Properties of log}$$

$$t = \frac{6}{\log 2} \approx 19.93 \qquad \text{Solve for } t$$

The bacteria level reaches 100,000 in about 20 hours.

Now Try Exercise 1 ■

EXAMPLE 2 ■ Rabbit Population

A certain breed of rabbit was introduced onto a small island 8 months ago. The current rabbit population on the island is estimated to be 4100 and doubling every 3 months.

(a) What was the initial size of the rabbit population?

(b) Estimate the population 1 year after the rabbits were introduced to the island.

(c) Sketch a graph of the rabbit population.

SOLUTION

(a) The doubling time is $a = 3$, so the population at time t is

$$n(t) = n_0 2^{t/3} \qquad \text{Model}$$

where n_0 is the initial population. Since the population is 4100 when t is 8 months, we have

$$n(8) = n_0 2^{8/3} \qquad \text{From model}$$

$$4100 = n_0 2^{8/3} \qquad \text{Because } n(8) = 4100$$

$$n_0 = \frac{4100}{2^{8/3}} \qquad \text{Divide by } 2^{8/3} \text{ and switch sides}$$

$$n_0 \approx 645 \qquad \text{Calculator}$$

Thus we estimate that 645 rabbits were introduced onto the island.

(b) From part (a) we know that the initial population is $n_0 = 645$, so we can model the population after t months by

$$n(t) = 645 \cdot 2^{t/3} \qquad \text{Model}$$

After 1 year $t = 12$, so

$$n(12) = 645 \cdot 2^{12/3} = 10{,}320$$

So after 1 year there would be about 10,000 rabbits.

(c) We first note that the domain is $t \geq 0$. The graph is shown in Figure 2.

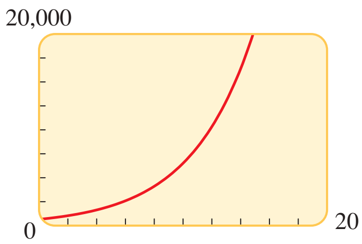


FIGURE 2 $n(t) = 645 \cdot 2^{t/3}$

Now Try Exercise 3 ■

■ Exponential Growth (Relative Growth Rate)

We have used an exponential function with base 2 to model population growth (in terms of the doubling time). We could also model the same population with an exponential function with base 3 (in terms of the tripling time). In fact, we can find an exponential model with any base. If we use the base e, we get a population model in terms of the **relative growth rate** r: the rate of population growth expressed as a proportion of the population at any time. In this case r is the "instantaneous" growth rate. (In calculus the concept of instantaneous rate is given a precise meaning.) For instance, if $r = 0.02$, then at any time t the growth rate is 2% of the population at time t.

The growth of a population with relative growth rate r is analogous to the growth of an investment with continuously compounded interest rate r.

EXPONENTIAL GROWTH (RELATIVE GROWTH RATE)

A population that experiences **exponential growth** increases according to the model

$$n(t) = n_0 e^{rt}$$

where $n(t)$ = population at time t

n_0 = initial size of the population

r = relative rate of growth (expressed as a proportion of the population)

t = time

Notice that the formula for population growth is the same as that for continuously compounded interest. In fact, the same principle is at work in both cases: The growth of a population (or an investment) per time period is proportional to the size of the population (or the amount of the investment). A population of 1,000,000 will increase more in one year than a population of 1000; in exactly the same way, an investment of \$1,000,000 will increase more in one year than an investment of \$1000.

In the following examples we assume that the populations grow exponentially.

EXAMPLE 3 ■ Predicting the Size of a Population

The initial bacterium count in a culture is 500. A biologist later makes a sample count of bacteria in the culture and finds that the relative rate of growth is 40% per hour.

(a) Find a function that models the number of bacteria after t hours.

(b) What is the estimated count after 10 hours?

(c) After how many hours will the bacteria count reach 80,000?

(d) Sketch a graph of the function $n(t)$.

SOLUTION

(a) We use the exponential growth model with $n_0 = 500$ and $r = 0.4$ to get

$$n(t) = 500e^{0.4t}$$

where t is measured in hours.

(b) Using the function in part (a), we find that the bacterium count after 10 hours is

$$n(10) = 500e^{0.4(10)} = 500e^4 \approx 27{,}300$$

(c) We set $n(t) = 80{,}000$ and solve the resulting exponential equation for t.

$$80{,}000 = 500 \cdot e^{0.4t} \qquad n(t) = 500 \cdot e^{0.4t}$$

$$160 = e^{0.4t} \qquad \text{Divide by 500}$$

$$\ln 160 = 0.4t \qquad \text{Take ln of each side}$$

$$t = \frac{\ln 160}{0.4} \approx 12.68 \qquad \text{Solve for } t$$

The bacteria level reaches 80,000 in about 12.7 hours.

(d) The graph is shown in Figure 3.

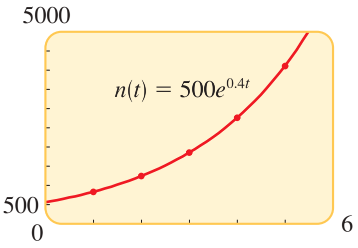


FIGURE 3

Now Try Exercise 5

EXAMPLE 4 ■ Comparing Different Rates of Population Growth

The relative growth of world population has been declining over the past few decades—from 2% in 1995 to 1.1% in 2013.

In 2000 the population of the world was 6.1 billion, and the relative rate of growth was 1.4% per year. It is claimed that a rate of 1.0% per year would make a significant difference in the total population in just a few decades. Test this claim by estimating the population of the world in the year 2050 using a relative rate of growth of (a) 1.4% per year and (b) 1.0% per year.

Graph the population functions for the next 100 years for the two relative growth rates in the same viewing rectangle.

Standing Room Only

The population of the world was about 6.1 billion in 2000 and was increasing at 1.4% per year. Assuming that each person occupies an average of 4 ft^2 of the surface of the earth, the exponential model for population growth projects that by the year 2801 there will be standing room only! (The total land surface area of the world is about 1.8×10^{15} ft^2.)

SOLUTION

(a) By the exponential growth model we have

$$n(t) = 6.1e^{0.014t}$$

where $n(t)$ is measured in billions and t is measured in years since 2000. Because the year 2050 is 50 years after 2000, we find

$$n(50) = 6.1e^{0.014(50)} = 6.1e^{0.7} \approx 12.3$$

The estimated population in the year 2050 is about 12.3 billion.

(b) We use the function

$$n(t) = 6.1e^{0.010t}$$

and find

$$n(50) = 6.1e^{0.010(50)} = 6.1e^{0.50} \approx 10.1$$

The estimated population in the year 2050 is about 10.1 billion.

The graphs in Figure 4 show that a small change in the relative rate of growth will, over time, make a large difference in population size.

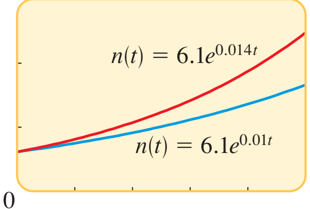


FIGURE 4

Now Try Exercise 7

EXAMPLE 5 ■ Expressing a Model in Terms of *e*

A culture starts with 10,000 bacteria, and the number doubles every 40 minutes.

(a) Find a function $n(t) = n_0 2^{t/a}$ that models the number of bacteria after t hours.

(b) Find a function $n(t) = n_0 e^{rt}$ that models the number of bacteria after t hours.

(c) Sketch a graph of the number of bacteria at time t.

SOLUTION

(a) The initial population is $n_0 = 10{,}000$. The doubling time is $a = 40$ min $= 2/3$ h. Since $1/a = 3/2 = 1.5$, the model is

$$n(t) = 10{,}000 \cdot 2^{1.5t}$$

(b) The initial population is $n_0 = 10{,}000$. We need to find the relative growth rate r. Since there are 20,000 bacteria when $t = 2/3$ h, we have

$$20{,}000 = 10{,}000e^{r(2/3)} \qquad n(t) = 10{,}000e^{rt}$$

$$2 = e^{r(2/3)} \qquad \text{Divide by 10,000}$$

$$\ln 2 = \ln e^{r(2/3)} \qquad \text{Take ln of each side}$$

$$\ln 2 = r(2/3) \qquad \text{Property of ln}$$

$$r = \frac{3 \ln 2}{2} \approx 1.0397 \qquad \text{Solve for } r$$

Now that we know the relative growth rate r, we can find the model:

$$n(t) = 10{,}000e^{1.0397t}$$

(c) We can graph the model in part (a) or the one in part (b). The graphs are identical. See Figure 5.

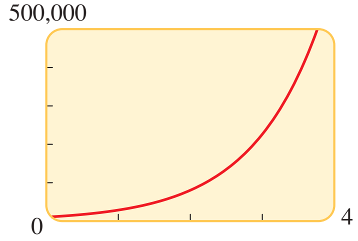


FIGURE 5 Graphs of $y = 10{,}000 \cdot 2^{1.5t}$ and $y = 10{,}000e^{1.0397t}$

Now Try Exercise 9

Radioactive Decay

The half-lives of **radioactive elements** vary from very long to very short. Here are some examples.

Element	Half-life
Thorium-232	14.5 billion years
Uranium-235	4.5 billion years
Thorium-230	80,000 years
Plutonium-239	24,360 years
Carbon-14	5,730 years
Radium-226	1,600 years
Cesium-137	30 years
Strontium-90	28 years
Polonium-210	140 days
Thorium-234	25 days
Iodine-135	8 days
Radon-222	3.8 days
Lead-211	3.6 minutes
Krypton-91	10 seconds

Radioactive substances decay by spontaneously emitting radiation. The rate of decay is proportional to the mass of the substance. This is analogous to population growth except that the mass *decreases*. Physicists express the rate of decay in terms of **half-life**, the time it takes for a sample of the substance to decay to half its original mass. For example, the half-life of radium-226 is 1600 years, so a 100-g sample decays to 50 g $\left(\text{or } \frac{1}{2} \times 100 \text{ g}\right)$ in 1600 years, then to 25 g $\left(\text{or } \frac{1}{2} \times \frac{1}{2} \times 100 \text{ g}\right)$ in 3200 years, and so on. In general, for a radioactive substance with mass m_0 and half-life h, the amount remaining at time t is modeled by

$$m(t) = m_0 2^{-t/h}$$

where h and t are measured in the same time units (minutes, hours, days, years, and so on).

To express this model in the form $m(t) = m_0 e^{rt}$, we need to find the relative decay rate r. Since h is the half-life, we have

$$m(t) = m_0 e^{-rt} \qquad \text{Model}$$

$$\frac{m_0}{2} = m_0 e^{-rh} \qquad h \text{ is the half-life}$$

$$\frac{1}{2} = e^{-rh} \qquad \text{Divide by } m_0$$

$$\ln \frac{1}{2} = -rh \qquad \text{Take ln of each side}$$

$$r = \frac{\ln 2}{h} \qquad \text{Solve for } r$$

This last equation allows us to find the relative decay rate r from the half-life h.

DISCOVERY PROJECT

Modeling Radiation with Coins and Dice

Radioactive elements decay when their atoms spontaneously emit radiation and change into smaller, stable atoms. But if atoms decay randomly, how is it possible to find a function that models their behavior? We'll try to answer this question by experimenting with randomly tossing coins and rolling dice. The experiments allow us to experience how a very large number of random events can result in predictable exponential results. You can find the project at **www.stewartmath.com**.

RADIOACTIVE DECAY MODEL

If m_0 is the initial mass of a radioactive substance with half-life h, then the mass remaining at time t is modeled by the function

$$m(t) = m_0 e^{-rt}$$

where $r = \dfrac{\ln 2}{h}$ is the **relative decay rate**.

EXAMPLE 6 ■ Radioactive Decay

Polonium-210 (^{210}Po) has a half-life of 140 days. Suppose a sample of this substance has a mass of 300 mg.

(a) Find a function $m(t) = m_0 2^{-t/h}$ that models the mass remaining after t days.

(b) Find a function $m(t) = m_0 e^{-rt}$ that models the mass remaining after t days.

(c) Find the mass remaining after one year.

(d) How long will it take for the sample to decay to a mass of 200 mg?

(e) Draw a graph of the sample mass as a function of time.

SOLUTION

(a) We have $m_0 = 300$ and $h = 140$, so the amount remaining after t days is

$$m(t) = 300 \cdot 2^{-t/140}$$

(b) We have $m_0 = 300$ and $r = \ln 2/140 \approx -0.00495$, so the amount remaining after t days is

$$m(t) = 300 \cdot e^{-0.00495t}$$

In parts (c) and (d) we can also use the model found in part (a). Check that the result is the same using either model.

(c) We use the function we found in part (a) with $t = 365$ (1 year):

$$m(365) = 300e^{-0.00495(365)} \approx 49.256$$

Thus approximately 49 mg of ^{210}Po remains after 1 year.

(d) We use the function that we found in part (b) with $m(t) = 200$ and solve the resulting exponential equation for t:

$300e^{-0.00495t} = 200$	$m(t) = m_0 e^{-rt}$
$e^{-0.00495t} = \frac{2}{3}$	Divide by 300
$\ln e^{-0.00495t} = \ln \frac{2}{3}$	Take ln of each side
$-0.00495t = \ln \frac{2}{3}$	Property of ln
$t = -\dfrac{\ln \frac{2}{3}}{0.00495}$	Solve for t
$t \approx 81.9$	Calculator

The time required for the sample to decay to 200 mg is about 82 days.

(e) We can graph the model in part (a) or the one in part (b). The graphs are identical. See Figure 6.

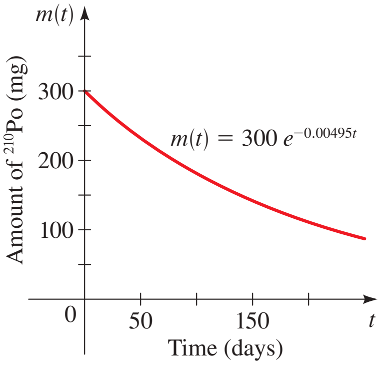


FIGURE 6

Now Try Exercise 17 ■


Joel W. Rogers/Encyclopedia/Corbis

Radioactive Waste

Harmful radioactive isotopes are produced whenever a nuclear reaction occurs, whether as the result of an atomic bomb test, a nuclear accident such as the one at Fukushima Daiichi in 2011, or the uneventful production of electricity at a nuclear power plant.

One radioactive material that is produced in atomic bombs is the isotope strontium-90 (^{90}Sr), with a half-life of 28 years. This is deposited like calcium in human bone tissue, where it can cause leukemia and other cancers. However, in the decades since atmospheric testing of nuclear weapons was halted, ^{90}Sr levels in the environment have fallen to a level that no longer poses a threat to health.

Nuclear power plants produce radioactive plutonium-239 (^{239}Pu), which has a half-life of 24,360 years. Because of its long half-life, ^{239}Pu could pose a threat to the environment for thousands of years. So great care must be taken to dispose of it properly. The difficulty of ensuring the safety of the disposed radioactive waste is one reason that nuclear power plants remain controversial.

■ Newton's Law of Cooling

Newton's Law of Cooling states that the rate at which an object cools is proportional to the temperature difference between the object and its surroundings, provided that the temperature difference is not too large. By using calculus, the following model can be deduced from this law.

NEWTON'S LAW OF COOLING

If D_0 is the initial temperature difference between an object and its surroundings, and if its surroundings have temperature T_s, then the temperature of the object at time t is modeled by the function

$$T(t) = T_s + D_0 e^{-kt}$$

where k is a positive constant that depends on the type of object.

EXAMPLE 7 ■ Newton's Law of Cooling

A cup of coffee has a temperature of 200°F and is placed in a room that has a temperature of 70°F. After 10 min the temperature of the coffee is 150°F.

(a) Find a function that models the temperature of the coffee at time t.

(b) Find the temperature of the coffee after 15 min.

(c) After how long will the coffee have cooled to 100°F?

(d) Illustrate by drawing a graph of the temperature function.

SOLUTION

(a) The temperature of the room is $T_s = 70°F$, and the initial temperature difference is

$$D_0 = 200 - 70 = 130°F$$

So by Newton's Law of Cooling, the temperature after t minutes is modeled by the function

$$T(t) = 70 + 130e^{-kt}$$

We need to find the constant k associated with this cup of coffee. To do this, we use the fact that when $t = 10$, the temperature is $T(10) = 150$. So we have

$70 + 130e^{-10k} = 150$	$T_s + D_0e^{-kt} = T(t)$
$130e^{-10k} = 80$	Subtract 70
$e^{-10k} = \frac{8}{13}$	Divide by 130
$-10k = \ln \frac{8}{13}$	Take ln of each side
$k = -\frac{1}{10} \ln \frac{8}{13}$	Solve for k
$k \approx 0.04855$	Calculator

Substituting this value of k into the expression for $T(t)$, we get

$$T(t) = 70 + 130e^{-0.04855t}$$

(b) We use the function that we found in part (a) with $t = 15$.

$$T(15) = 70 + 130e^{-0.04855(15)} \approx 133°F$$

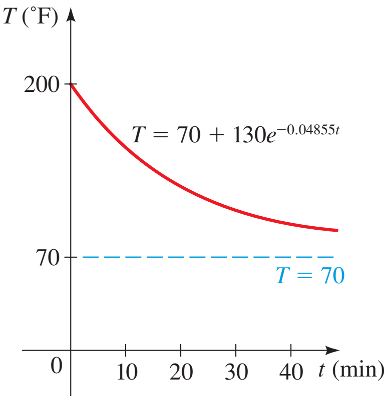


FIGURE 7 Temperature of coffee after t minutes

(c) We use the function that we found in part (a) with $T(t) = 100$ and solve the resulting exponential equation for t.

$$70 + 130e^{-0.04855t} = 100 \qquad T_s + D_0e^{-kt} = T(t)$$
$$130e^{-0.04855t} = 30 \qquad \text{Subtract 70}$$
$$e^{-0.04855t} = \tfrac{3}{13} \qquad \text{Divide by 130}$$
$$-0.04855t = \ln \tfrac{3}{13} \qquad \text{Take ln of each side}$$
$$t = \frac{\ln \frac{3}{13}}{-0.04855} \qquad \text{Solve for } t$$
$$t \approx 30.2 \qquad \text{Calculator}$$

The coffee will have cooled to 100°F after about half an hour.

(d) The graph of the temperature function is sketched in Figure 7. Notice that the line $t = 70$ is a horizontal asymptote. (Why?)

Now Try Exercise 25 ■

4.6 EXERCISES

APPLICATIONS

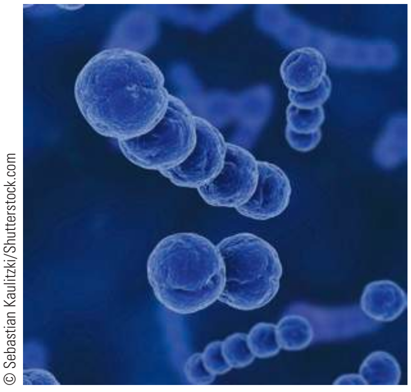
1–16 ■ Population Growth These exercises use the population growth model.

1. Bacteria Culture A certain culture of the bacterium *Streptococcus A* initially has 10 bacteria and is observed to double every 1.5 hours.

(a) Find an exponential model $n(t) = n_0 2^{t/a}$ for the number of bacteria in the culture after t hours.

(b) Estimate the number of bacteria after 35 hours.

(c) After how many hours will the bacteria count reach 10,000?



Streptococcus A
(12,000 × magnification)

2. Bacteria Culture A certain culture of the bacterium *Rhodobacter sphaeroides* initially has 25 bacteria and is observed to double every 5 hours.

(a) Find an exponential model $n(t) = n_0 2^{t/a}$ for the number of bacteria in the culture after t hours.

(b) Estimate the number of bacteria after 18 hours.

(c) After how many hours will the bacteria count reach 1 million?

3. Squirrel Population A grey squirrel population was introduced in a certain county of Great Britain 30 years ago. Biologists observe that the population doubles every 6 years, and now the population is 100,000.

(a) What was the initial size of the squirrel population?

(b) Estimate the squirrel population 10 years from now.

(c) Sketch a graph of the squirrel population.

4. Bird Population A certain species of bird was introduced in a certain county 25 years ago. Biologists observe that the population doubles every 10 years, and now the population is 13,000.

(a) What was the initial size of the bird population?

(b) Estimate the bird population 5 years from now.

(c) Sketch a graph of the bird population.

5. Fox Population The fox population in a certain region has a relative growth rate of 8% per year. It is estimated that the population in 2013 was 18,000.

(a) Find a function $n(t) = n_0 e^{rt}$ that models the population t years after 2013.

(b) Use the function from part (a) to estimate the fox population in the year 2021.

(c) After how many years will the fox population reach 25,000?

(d) Sketch a graph of the fox population function for the years 2013–2021.

6. Fish Population The population of a certain species of fish has a relative growth rate of 1.2% per year. It is estimated that the population in 2010 was 12 million.

(a) Find an exponential model $n(t) = n_0 e^{rt}$ for the population t years after 2010.

(b) Estimate the fish population in the year 2015.

(c) After how many years will the fish population reach 14 million?

(d) Sketch a graph of the fish population.

7. Population of a Country The population of a country has a relative growth rate of 3% per year. The government is trying to reduce the growth rate to 2%. The population in 2011 was approximately 110 million. Find the projected population for the year 2036 for the following conditions.

(a) The relative growth rate remains at 3% per year.

(b) The relative growth rate is reduced to 2% per year.

8. Bacteria Culture It is observed that a certain bacteria culture has a relative growth rate of 12% per hour, but in the presence of an antibiotic the relative growth rate is reduced to 5% per hour. The initial number of bacteria in the culture is 22. Find the projected population after 24 hours for the following conditions.

(a) No antibiotic is present, so the relative growth rate is 12%.

(b) An antibiotic is present in the culture, so the relative growth rate is reduced to 5%.

9. Population of a City The population of a certain city was 112,000 in 2014, and the observed doubling time for the population is 18 years.

(a) Find an exponential model $n(t) = n_0 2^{t/a}$ for the population t years after 2014.

(b) Find an exponential model $n(t) = n_0 e^{rt}$ for the population t years after 2014.

(c) Sketch a graph of the population at time t.

(d) Estimate how long it takes the population to reach 500,000.

10. Bat Population The bat population in a certain Midwestern county was 350,000 in 2012, and the observed doubling time for the population is 25 years.

(a) Find an exponential model $n(t) = n_0 2^{t/a}$ for the population t years after 2012.

(b) Find an exponential model $n(t) = n_0 e^{rt}$ for the population t years after 2012.

(c) Sketch a graph of the population at time t.

(d) Estimate how long it takes the population to reach 2 million.

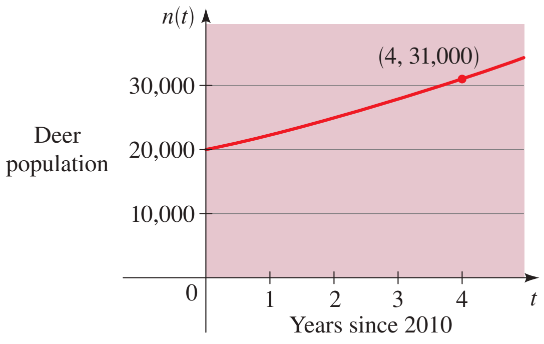
11. Deer Population The graph shows the deer population in a Pennsylvania county between 2010 and 2014. Assume that the population grows exponentially.

(a) What was the deer population in 2010?

(b) Find a function that models the deer population t years after 2010.

(c) What is the projected deer population in 2018?

(d) Estimate how long it takes the population to reach 100,000.



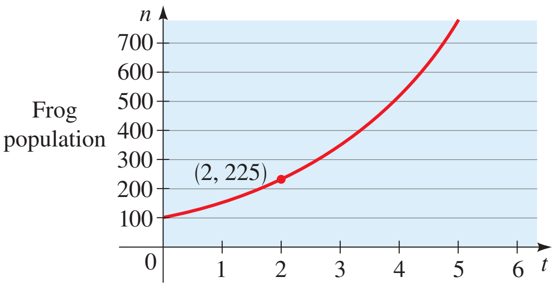
12. Frog Population Some bullfrogs were introduced into a small pond. The graph shows the bullfrog population for the next few years. Assume that the population grows exponentially.

(a) What was the initial bullfrog population?

(b) Find a function that models the bullfrog population t years since the bullfrogs were put into the pond.

(c) What is the projected bullfrog population after 15 years?

(d) Estimate how long it takes the population to reach 75,000.



13. Bacteria Culture A culture starts with 8600 bacteria. After 1 hour the count is 10,000.

(a) Find a function that models the number of bacteria $n(t)$ after t hours.

(b) Find the number of bacteria after 2 hours.

(c) After how many hours will the number of bacteria double?

14. Bacteria Culture The count in a culture of bacteria was 400 after 2 hours and 25,600 after 6 hours.

(a) What is the relative rate of growth of the bacteria population? Express your answer as a percentage.

(b) What was the initial size of the culture?

(c) Find a function that models the number of bacteria $n(t)$ after t hours.

(d) Find the number of bacteria after 4.5 hours.

(e) After how many hours will the number of bacteria reach 50,000?

15. Population of California The population of California was 29.76 million in 1990 and 33.87 million in 2000. Assume that the population grows exponentially.

(a) Find a function that models the population t years after 1990.

(b) Find the time required for the population to double.

(c) Use the function from part (a) to predict the population of California in the year 2010. Look up California's actual population in 2010, and compare.

16. World Population The population of the world was 7.1 billion in 2013, and the observed relative growth rate was 1.1% per year.

(a) Estimate how long it takes the population to double.

(b) Estimate how long it takes the population to triple.

17–24 ■ Radioactive Decay These exercises use the radioactive decay model.

17. Radioactive Radium The half-life of radium-226 is 1600 years. Suppose we have a 22-mg sample.

(a) Find a function $m(t) = m_0 2^{-t/h}$ that models the mass remaining after t years.

(b) Find a function $m(t) = m_0 e^{-rt}$ that models the mass remaining after t years.

(c) How much of the sample will remain after 4000 years?

(d) After how many years will only 18 mg of the sample remain?

18. Radioactive Cesium The half-life of cesium-137 is 30 years. Suppose we have a 10-g sample.

(a) Find a function $m(t) = m_0 2^{-t/h}$ that models the mass remaining after t years.

(b) Find a function $m(t) = m_0 e^{-rt}$ that models the mass remaining after t years.

(c) How much of the sample will remain after 80 years?

(d) After how many years will only 2 g of the sample remain?

19. Radioactive Strontium The half-life of strontium-90 is 28 years. How long will it take a 50-mg sample to decay to a mass of 32 mg?

20. Radioactive Radium Radium-221 has a half-life of 30 s. How long will it take for 95% of a sample to decay?

21. Finding Half-Life If 250 mg of a radioactive element decays to 200 mg in 48 hours, find the half-life of the element.

22. Radioactive Radon After 3 days a sample of radon-222 has decayed to 58% of its original amount.

(a) What is the half-life of radon-222?

(b) How long will it take the sample to decay to 20% of its original amount?

23. Carbon-14 Dating A wooden artifact from an ancient tomb contains 65% of the carbon-14 that is present in living trees. How long ago was the artifact made? (The half-life of carbon-14 is 5730 years.)

24. Carbon-14 Dating The burial cloth of an Egyptian mummy is estimated to contain 59% of the carbon-14 it contained originally. How long ago was the mummy buried? (The half-life of carbon-14 is 5730 years.)

25–28 ■ Law of Cooling These exercises use Newton's Law of Cooling.

25. Cooling Soup A hot bowl of soup is served at a dinner party. It starts to cool according to Newton's Law of Cooling, so its temperature at time t is given by

$$T(t) = 65 + 145e^{-0.05t}$$

where t is measured in minutes and T is measured in °F.

(a) What is the initial temperature of the soup?

(b) What is the temperature after 10 min?

(c) After how long will the temperature be 100°F?

26. Time of Death Newton's Law of Cooling is used in homicide investigations to determine the time of death. The normal body temperature is 98.6°F. Immediately following death, the body begins to cool. It has been determined experimentally that the constant in Newton's Law of Cooling is approximately $k = 0.1947$, assuming that time is measured in hours. Suppose that the temperature of the surroundings is 60°F.

(a) Find a function $T(t)$ that models the temperature t hours after death.

(b) If the temperature of the body is now 72°F, how long ago was the time of death?

27. Cooling Turkey A roasted turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F.

(a) If the temperature of the turkey is 150°F after half an hour, what is its temperature after 45 min?

(b) After how many hours will the turkey cool to 100°F?

28. Boiling Water A kettle full of water is brought to a boil in a room with temperature 20°C. After 15 min the temperature of the water has decreased from 100°C to 75°C. Find the temperature after another 10 min. Illustrate by graphing the temperature function.

4.7 LOGARITHMIC SCALES

■ The pH Scale ■ The Richter Scale ■ The Decibel Scale

Animal	W (kg)	$\log W$
Ant	0.000003	−5.5
Elephant	4000	3.6
Whale	170,000	5.2

When a physical quantity varies over a very large range, it is often convenient to take its logarithm in order to work with more manageable numbers. On a **logarithmic scale**, numbers are represented by their logarithms. For example, the table in the margin gives the weights W of some animals (in kilograms) and their logarithms ($\log W$).

The weights (W) vary enormously, but on a logarithmic scale, the weights are represented by more manageable numbers ($\log W$). Figure 1 shows that it is difficult to compare the weights W graphically but easy to compare them on a logarithmic scale.

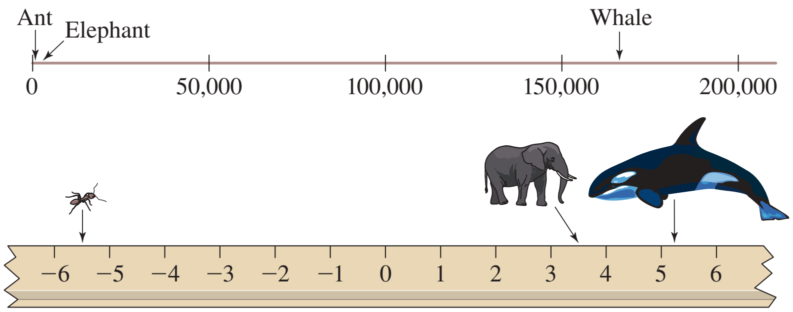


FIGURE 1 Weight graphed on the real line (top) and on a logarithmic scale (bottom)

We discuss three commonly used logarithmic scales: the pH scale, which measures acidity; the Richter scale, which measures the intensity of earthquakes; and the decibel scale, which measures the loudness of sounds. Other quantities that are measured on logarithmic scales are light intensity, information capacity, and radiation.

■ The pH Scale

Chemists measured the acidity of a solution by giving its hydrogen ion concentration until Søren Peter Lauritz Sørensen, in 1909, proposed a more convenient measure. He defined

$$\text{pH} = -\log[\text{H}^+]$$

where $[\text{H}^+]$ is the concentration of hydrogen ions measured in moles per liter (M). He did this to avoid very small numbers and negative exponents. For instance,

$$\text{if} \quad [\text{H}^+] = 10^{-4}\text{ M}, \quad \text{then} \quad \text{pH} = -\log_{10}(10^{-4}) = -(-4) = 4$$

Solutions with a pH of 7 are defined as *neutral*, those with pH < 7 are *acidic*, and those with pH > 7 are *basic*. Notice that when the pH increases by one unit, $[\text{H}^+]$ decreases by a factor of 10.

pH for Some Common Substances

Substance	pH
Milk of magnesia	10.5
Seawater	8.0–8.4
Human blood	7.3–7.5
Crackers	7.0–8.5
Hominy	6.9–7.9
Cow's milk	6.4–6.8
Spinach	5.1–5.7
Tomatoes	4.1–4.4
Oranges	3.0–4.0
Apples	2.9–3.3
Limes	1.3–2.0
Battery acid	1.0

EXAMPLE 1 ■ pH Scale and Hydrogen Ion Concentration

(a) The hydrogen ion concentration of a sample of human blood was measured to be $[\text{H}^+] = 3.16 \times 10^{-8}$ M. Find the pH, and classify the blood as acidic or basic.

(b) The most acidic rainfall ever measured occurred in Scotland in 1974; its pH was 2.4. Find the hydrogen ion concentration.

SOLUTION

(a) A calculator gives

$$\text{pH} = -\log[\text{H}^+] = -\log(3.16 \times 10^{-8}) \approx 7.5$$

Since this is greater than 7, the blood is basic.

(b) To find the hydrogen ion concentration, we need to solve for $[\text{H}^+]$ in the logarithmic equation

$$\log[\text{H}^+] = -\text{pH}$$

So we write it in exponential form:

$$[\text{H}^+] = 10^{-\text{pH}}$$

In this case pH = 2.4, so

$$[\text{H}^+] = 10^{-2.4} \approx 4.0 \times 10^{-3}\ \text{M}$$

Now Try Exercises 1 and 3 ■

The Richter Scale

In 1935 the American geologist Charles Richter (1900–1984) defined the magnitude M of an earthquake to be

$$M = \log \frac{I}{S}$$

where I is the intensity of the earthquake (measured by the amplitude of a seismograph reading taken 100 km from the epicenter of the earthquake) and S is the intensity of a "standard" earthquake (whose amplitude is 1 micron $= 10^{-4}$ cm). (In practice, seismograph stations may not be exactly 100 km from the epicenter, so appropriate adjustments are made in calculating the magnitude of an earthquake.) The magnitude of a standard earthquake is

$$M = \log \frac{S}{S} = \log 1 = 0$$

Richter studied many earthquakes that occurred between 1900 and 1950. The largest had magnitude 8.9 on the Richter scale, and the smallest had magnitude 0. This corresponds to a ratio of intensities of 800,000,000, so the Richter scale provides more

Largest Earthquakes

Location	Date	Magnitude
Chile	1960	9.5
Alaska	1964	9.2
Japan	2011	9.1
Sumatra	2004	9.1
Kamchatka	1952	9.0
Chile	2010	8.8
Ecuador	1906	8.8
Alaska	1965	8.7
Alaska	1957	8.6
Sumatra	2005	8.6
Sumatra	2012	8.6
Tibet	1950	8.6
Indonesia	1938	8.5
Kamchatka	1923	8.5

Source: U.S. Geological Society


Robert Vos/AFP/Getty Images

DISCOVERY PROJECT

The Even-Tempered Clavier

Poets, writers, philosophers, and even politicians have extolled the virtues of music—its beauty and its power to communicate emotion. But at the heart of music is a logarithmic scale. The tones that we are familiar with from our everyday listening can all be reproduced by the keys of a piano. The keys of a piano, in turn, are "evenly tempered" using a logarithmic scale. In this project we explore how exponential and logarithmic functions are used in properly tuning a piano. You can find the project at **www.stewartmath.com**.

manageable numbers to work with. For instance, an earthquake of magnitude 6 is ten times stronger than an earthquake of magnitude 5.

EXAMPLE 2 ■ Magnitude and Intensity

(a) Find the magnitude of an earthquake that has an intensity of 3.75 (that is, the amplitude of the seismograph reading is 3.75 cm).

(b) An earthquake was measured to have a magnitude of 5.1 on the Richter scale. Find the intensity of the earthquake.

SOLUTION

(a) From the definition of magnitude we see that

$$M = \log \frac{I}{S} = \log \frac{3.75}{10^{-4}} = \log 37500 \approx 4.6$$

Thus the magnitude is 4.6 on the Richter scale.

(b) To find the intensity, we need to solve for I in the logarithmic equation

$$M = \log \frac{I}{S}$$

So we write it in exponential form:

$$10^M = \frac{I}{S}$$

In this case $S = 10^{-4}$ and $M = 5.1$, so

$$10^{5.1} = \frac{I}{10^{-4}} \qquad M = 5.1, S = 10^{-4}$$

$$(10^{-4})(10^{5.1}) = I \qquad \text{Multiply by } 10^{-4}$$

$$I = 10^{1.1} \approx 12.6 \qquad \text{Add exponents}$$

Thus the intensity of the earthquake is about 12.6, which means that the amplitude of the seismograph reading is about 12.6 cm.

Now Try Exercise 9 ■

EXAMPLE 3 ■ Magnitude of Earthquakes

There are several other logarithmic scales used to calculate the magnitude of earthquakes. For instance, the U.S. Geological Survey uses the *moment magnitude scale*.

The 1906 earthquake in San Francisco had an estimated magnitude of 8.3 on the Richter scale. In the same year a powerful earthquake occurred on the Colombia-Ecuador border that was four times as intense. What was the magnitude of the Colombia-Ecuador earthquake on the Richter scale?

SOLUTION If I is the intensity of the San Francisco earthquake, then from the definition of magnitude we have

$$M = \log \frac{I}{S} = 8.3$$

The intensity of the Colombia-Ecuador earthquake was $4I$, so its magnitude was

$$M = \log \frac{4I}{S} = \log 4 + \log \frac{I}{S} = \log 4 + 8.3 \approx 8.9$$

Now Try Exercise 11 ■

EXAMPLE 4 ■ Intensity of Earthquakes

Grant Smith/Corbis

The 1989 Loma Prieta earthquake that shook San Francisco had a magnitude of 7.1 on the Richter scale. How many times more intense was the 1906 earthquake (see Example 3) than the 1989 event?

SOLUTION If I_1 and I_2 are the intensities of the 1906 and 1989 earthquakes, then we are required to find I_1/I_2. To relate this to the definition of magnitude, we divide the numerator and denominator by S.

$$\log \frac{I_1}{I_2} = \log \frac{I_1/S}{I_2/S} \qquad \text{Divide numerator and denominator by } S$$

$$= \log \frac{I_1}{S} - \log \frac{I_2}{S} \qquad \text{Law 2 of logarithms}$$

$$= 8.3 - 7.1 = 1.2 \qquad \text{Definition of earthquake magnitude}$$

Therefore

$$\frac{I_1}{I_2} = 10^{\log(I_1/I_2)} = 10^{1.2} \approx 16$$

The 1906 earthquake was about 16 times as intense as the 1989 earthquake.

Now Try Exercise 13 ■

■ The Decibel Scale

The ear is sensitive to an extremely wide range of sound intensities. We take as a reference intensity $I_0 = 10^{-12}$ W/m^2 (watts per square meter) at a frequency of 1000 hertz, which measures a sound that is just barely audible (the threshold of hearing). The psychological sensation of loudness varies with the logarithm of the intensity (the Weber-Fechner Law), so the **decibel level** B, measured in decibels (dB), is defined as

$$B = 10 \log \frac{I}{I_0}$$

The decibel level of the barely audible reference sound is

$$B = 10 \log \frac{I_0}{I_0} = 10 \log 1 = 0 \text{ dB}$$

EXAMPLE 5 ■ Decibel Level and Intensity

(a) Find the decibel level of a jet engine at takeoff if the intensity was measured at 100 W/m^2.

(b) Find the intensity level of a motorcycle engine at full throttle if the decibel level was measured at 90 dB.

SOLUTION

(a) From the definition of decibel level we see that

$$B = 10 \log \frac{I}{I_0} = 10 \log \frac{10^2}{10^{-12}} = 10 \log 10^{14} = 140 \text{ dB}$$

Thus the decibel level is 140 dB.

(b) To find the intensity, we need to solve for I in the logarithmic equation

$$B = 10 \log \frac{I}{I_0} \qquad \text{Definition of decibel level}$$

$$\frac{B}{10} = \log I - \log 10^{-12} \qquad \text{Divide by 10, } I_0 = 10^{-12}$$

$$\frac{B}{10} = \log I + 12 \qquad \text{Definition of logarithm}$$

$$\frac{B}{10} - 12 = \log I \qquad \text{Subtract 12}$$

$$\log I = \frac{90}{10} - 12 = -3 \qquad B = 90$$

$$I = 10^{-3} \qquad \text{Exponential form}$$

Thus the intensity is 10^{-3} W/m^2.

Now Try Exercises 15 and 17 ■

The **decibel levels of sounds** that we can hear vary from very loud to very soft. Here are some examples of the decibel levels of commonly heard sounds.

Source of sound	B (dB)
Jet takeoff	140
Jackhammer	130
Rock concert	120
Subway	100
Heavy traffic	80
Ordinary traffic	70
Normal conversation	50
Whisper	30
Rustling leaves	10–20
Threshold of hearing	0

The table in the margin lists decibel levels for some common sounds ranging from the threshold of human hearing to the jet takeoff of Example 5. The threshold of pain is about 120 dB.

4.7 EXERCISES

APPLICATIONS

1. **Finding pH** The hydrogen ion concentration of a sample of each substance is given. Calculate the pH of the substance.
   (a) Lemon juice: $[H^+] = 5.0 \times 10^{-3}$ M
   (b) Tomato juice: $[H^+] = 3.2 \times 10^{-4}$ M
   (c) Seawater: $[H^+] = 5.0 \times 10^{-9}$ M

2. **Finding pH** An unknown substance has a hydrogen ion concentration of $[H^+] = 3.1 \times 10^{-8}$ M. Find the pH and classify the substance as acidic or basic.

3. **Ion Concentration** The pH reading of a sample of each substance is given. Calculate the hydrogen ion concentration of the substance.
   (a) Vinegar: pH = 3.0 **(b)** Milk: pH = 6.5

4. **Ion Concentration** The pH reading of a glass of liquid is given. Find the hydrogen ion concentration of the liquid.
   (a) Beer: pH = 4.6 **(b)** Water: pH = 7.3

5. **Finding pH** The hydrogen ion concentrations in cheeses range from 4.0×10^{-7} M to 1.6×10^{-5} M. Find the corresponding range of pH readings.

6. **Ion Concentration in Wine** The pH readings for wines vary from 2.8 to 3.8. Find the corresponding range of hydrogen ion concentrations.

7. **pH of Wine** If the pH of a wine is too high, say, 4.0 or above, the wine becomes unstable and has a flat taste.
   (a) A certain California red wine has a pH of 3.2, and a certain Italian white wine has a pH of 2.9. Find the corresponding hydrogen ion concentrations of the two wines.
   (b) Which wine has the lower hydrogen ion concentration?

8. **pH of Saliva** The pH of saliva is normally in the range of 6.4 to 7.0. However, when a person is ill, the person's saliva becomes more acidic.
   (a) When Marco is sick, he tests the pH of his saliva and finds that it is 5.5. What is the hydrogen ion concentration of his saliva?
   (b) Will the hydrogen ion concentration in Marco's saliva increase or decrease as he gets better?
   (c) After Marco recovers, he tests the pH of his saliva, and it is 6.5. Was the saliva more acidic or less acidic when he was sick?

9. **Earthquake Magnitude and Intensity**
   (a) Find the magnitude of an earthquake that has an intensity that is 31.25 (that is, the amplitude of the seismograph reading is 31.25 cm).
   (b) An earthquake was measured to have a magnitude of 4.8 on the Richter scale. Find the intensity of the earthquake.

10. Earthquake Magnitude and Intensity

(a) Find the magnitude of an earthquake that has an intensity that is 72.1 (that is, the amplitude of the seismograph reading is 72.1 cm).

(b) An earthquake was measured to have a magnitude of 5.8 on the Richter scale. Find the intensity of the earthquake.

11. Earthquake Magnitudes If one earthquake is 20 times as intense as another, how much larger is its magnitude on the Richter scale?

12. Earthquake Magnitudes The 1906 earthquake in San Francisco had a magnitude of 8.3 on the Richter scale. At the same time in Japan an earthquake with magnitude 4.9 caused only minor damage. How many times more intense was the San Francisco earthquake than the Japan earthquake?

13. Earthquake Magnitudes The Japan earthquake of 2011 had a magnitude of 9.1 on the Richter scale. How many times more intense was this than the 1906 San Francisco earthquake? (See Exercise 12.)

14. Earthquake Magnitudes The Northridge, California, earthquake of 1994 had a magnitude of 6.8 on the Richter scale. A year later, a 7.2-magnitude earthquake struck Kobe, Japan. How many times more intense was the Kobe earthquake than the Northridge earthquake?

15. Traffic Noise The intensity of the sound of traffic at a busy intersection was measured at 2.0×10^{-5} W/m^2. Find the decibel level.

16. Leaf Blower The intensity of the sound from a certain leaf blower is measured at 3.2×10^{-2} W/m^2. Find the decibel level.

17. Hair Dryer The decibel level of the sound from a certain hair dryer is measured at 70 dB. Find the intensity of the sound.

18. Subway Noise The decibel level of the sound of a subway train was measured at 98 dB. Find the intensity in watts per square meter (W/m^2).

19. Hearing Loss from MP3 Players Recent research has shown that the use of earbud-style headphones packaged with MP3 players can cause permanent hearing loss.

(a) The intensity of the sound from the speakers of a certain MP3 player (without earbuds) is measured at 3.1×10^{-5} W/m^2. Find the decibel level.

(b) If earbuds are used with the MP3 player in part (a), the decibel level is 95 dB. Find the intensity.

(c) Find the ratio of the intensity of the sound from the MP3 player with earbuds to that of the sound without earbuds.

20. Comparing Decibel Levels The noise from a power mower was measured at 106 dB. The noise level at a rock concert was measured at 120 dB. Find the ratio of the intensity of the rock music to that of the power mower.

DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

21. PROVE: Inverse Square Law for Sound A law of physics states that the intensity of sound is inversely proportional to the square of the distance d from the source: $I = k/d^2$.

(a) Use this model and the equation

$$B = 10 \log \frac{I}{I_0}$$

(described in this section) to show that the decibel levels B_1 and B_2 at distances d_1 and d_2 from a sound source are related by the equation

$$B_2 = B_1 + 20 \log \frac{d_1}{d_2}$$

(b) The intensity level at a rock concert is 120 dB at a distance 2 m from the speakers. Find the intensity level at a distance of 10 m.

CHAPTER 4 ■ REVIEW

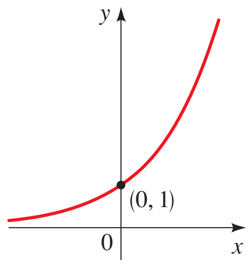
PROPERTIES AND FORMULAS

Exponential Functions (pp. 330–332)

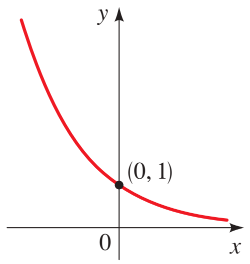
The **exponential function** f with base a (where $a > 0$, $a \neq 1$) is defined for all real numbers x by

$$f(x) = a^x$$

The domain of f is $\mathbb{R}$, and the range of f is $(0, \infty)$ The graph of f has one of the following shapes, depending on the value of a:



$f(x) = a^x$ for $a > 1$



$f(x) = a^x$ for $0 < a < 1$

The Natural Exponential Function (p. 339)

The **natural exponential function** is the exponential function with base e:

$$f(x) = e^x$$

The number e is defined to be the number that the expression $(1 + 1/n)^n$ approaches as $n \to \infty$. An approximate value for the irrational number e is

$$e \approx 2.7182818284590\ldots$$

Compound Interest (pp. 334, 340)

If a principal P is invested in an account paying an annual interest rate r, compounded n times a year, then after t years the **amount** $A(t)$ in the account is

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

If the interest is compounded **continuously**, then the amount is

$$A(t) = Pe^{rt}$$

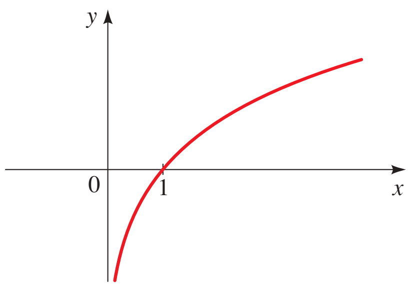
Logarithmic Functions (pp. 344–345)

The **logarithmic function** $\log_a$ with base a (where $a > 0, a \neq 1$) is defined for $x > 0$ by

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

So $\log_a x$ is the exponent to which the base a must be raised to give y.

The domain of $\log_a$ is $(0, \infty)$, and the range is $\mathbb{R}$. For $a > 1$, the graph of the function $\log_a$ has the following shape:



$y = \log_a x, a > 1$

Common and Natural Logarithms (pp. 348–349)

The logarithm function with base 10 is called the **common logarithm** and is denoted **log**. So

$$\log x = \log_{10} x$$

The logarithm function with base e is called the **natural logarithm** and is denoted **ln**. So

$$\ln x = \log_e x$$

Properties of Logarithms (pp. 345, 349)

1. $\log_a 1 = 0$

2. $\log_a a = 1$

3. $\log_a a^x = x$

4. $a^{\log_a x} = x$

Laws of Logarithms (p. 354)

Let a be a logarithm base $(a > 0, a \neq 1)$, and let A, B, and C be any real numbers or algebraic expressions that represent real numbers, with $A > 0$ and $B > 0$. Then:

1. $\log_a(AB) = \log_a A + \log_a B$

2. $\log_a(A/B) = \log_a A - \log_a B$

3. $\log_a(A^C) = C \log_a A$

Change of Base Formula (p. 357)

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Guidelines for Solving Exponential Equations (p. 361)

1. Isolate the exponential term on one side of the equation.

2. Take the logarithm of each side, and use the Laws of Logarithms to “bring down the exponent.”

3. Solve for the variable.

Guidelines for Solving Logarithmic Equations (p. 364)

1. Isolate the logarithmic term(s) on one side of the equation, and use the Laws of Logarithms to combine logarithmic terms if necessary.

2. Rewrite the equation in exponential form.

3. Solve for the variable.

Exponential Growth Model (p. 373)

A population experiences **exponential growth** if it can be modeled by the exponential function

$$n(t) = n_0 e^{rt}$$

where $n(t)$ is the population at time t, n_0 is the initial population (at time $t = 0$), and r is the relative growth rate (expressed as a proportion of the population).

Radioactive Decay Model (pp. 375–376)

If a **radioactive substance** with half-life h has initial mass m_0, then at time t the mass $m(t)$ of the substance that remains is modeled by the exponential function

$$m(t) = m_0 e^{-rt}$$

where $r = \dfrac{\ln 2}{h}$.

Newton’s Law of Cooling (p. 377)

If an object has an initial temperature that is D_0 degrees warmer than the surrounding temperature T_s, then at time t the temperature $T(t)$ of the object is modeled by the function

$$T(t) = T_s + D_0 e^{-kt}$$

where the constant $k > 0$ depends on the size and type of the object.

Logarithmic Scales (pp. 381–385)

The **pH scale** measures the acidity of a solution:

$$\text{pH} = -\log[\text{H}^+]$$

The **Richter scale** measures the intensity of earthquakes:

$$M = \log \frac{I}{S}$$

The **decibel scale** measures the intensity of sound:

$$B = 10 \log \frac{I}{I_0}$$

CONCEPT CHECK

1. Let f be the exponential function with base a.
   (a) Write an equation that defines f.
   (b) Write an equation for the exponential function f with base 3.

2. Let f be the exponential function $f(x) = a^x$, where $a > 0$.
   (a) What is the domain of f?
   (b) What is the range of f?
   (c) Sketch graphs of f for the following cases.
   (i) $a > 1$ (ii) $0 < a < 1$

3. If x is large, which function grows faster, $f(x) = 2^x$ or $g(x) = x^2$?

4. (a) How is the number e defined?
   (b) Give an approximate value of e, correct to four decimal places.
   (c) What is the natural exponential function?

5. (a) How is $\log_a x$ defined?
   (b) Find $\log_3 9$.
   (c) What is the natural logarithm?
   (d) What is the common logarithm?
   (e) Write the exponential form of the equation $\log_7 49 = 2$.

6. Let f be the logarithmic function $f(x) = \log_a x$.
   (a) What is the domain of f?
   (b) What is the range of f?
   (c) Sketch a graph of the logarithmic function for the case that $a > 1$.

7. State the three Laws of Logarithms.

8. (a) State the Change of Base Formula.
   (b) Find $\log_7 30$.

9. (a) What is an exponential equation?
   (b) How do you solve an exponential equation?
   (c) Solve for x: $2^x = 19$

10. (a) What is a logarithmic equation?
    (b) How do you solve a logarithmic equation?
    (c) Solve for x: $4 \log_3 x = 7$

11. Suppose that an amount P is invested at an interest rate r and $A(t)$ is the amount of the investment after t years. Write a formula for $A(t)$ in the following cases.
    (a) Interest is compounded n times per year.
    (b) Interest is compounded continuously.

12. Suppose that the initial size of a population is n_0 and the population grows exponentially. Let $n(t)$ be the size of the population at time t.
    (a) Write a formula for $n(t)$ in terms of the doubling time a.
    (b) Write a formula for $n(t)$ in terms of the relative growth rate r.

13. Suppose that the initial mass of a radioactive substance is m_0 and the half-life of the substance is h. Let $m(t)$ be the mass remaining at time t.
    (a) What is meant by the half-life h?
    (b) Write a formula for $m(t)$ in terms of the half-life h.
    (c) Write a formula for the relative decay rate r in terms of the half-life h.
    (d) Write a formula for $m(t)$ in terms of the relative decay rate r.

14. Suppose that the initial temperature difference between an object and its surroundings is D_0 and the surroundings have temperature T_s. Let $T(t)$ be the temperature at time t. State Newton's Law of Cooling for $T(t)$.

15. What is a logarithmic scale? If we use a logarithmic scale with base 10, what do the following numbers correspond to on the logarithmic scale?
    (i) 100 (ii) 100,000 (iii) 0.0001

16. (a) What does the pH scale measure?
    (b) Define the pH of a substance with hydrogen ion concentration of $[H^+]$.

17. (a) What does the Richter scale measure?
    (b) Define the magnitude M of an earthquake in terms of the intensity I of the earthquake and the intensity S of a standard earthquake.

18. (a) What does the decibel scale measure?
    (b) Define the decibel level B of a sound in terms of the intensity I of the sound and the intensity I_0 of a barely audible sound.

ANSWERS TO THE CONCEPT CHECK CAN BE FOUND AT THE BACK OF THE BOOK.

EXERCISES

1–4 ■ Evaluating Exponential Functions Use a calculator to find the indicated values of the exponential function, rounded to three decimal places.

1. $f(x) = 5^x$; $f(-1.5), f(\sqrt{2}), f(2.5)$
2. $f(x) = 3 \cdot 2^x$; $f(-2.2), f(\sqrt{7}), f(5.5)$
3. $g(x) = 4e^{x-2}$; $g(-0.7), g(1), g(\pi)$
4. $g(x) = \frac{7}{4}e^{x+1}$; $g(-2), g(\sqrt{3}), g(3.6)$

5–16 ■ Graphing Exponential and Logarithmic Functions Sketch the graph of the function. State the domain, range, and asymptote.

5. $f(x) = 3^{x-2}$
6. $f(x) = 2^{-x+1}$
7. $g(x) = 3 + 2^x$
8. $g(x) = 5^{-x} - 5$
9. $F(x) = e^{x-1} + 1$
10. $G(x) = -e^{x+1} - 2$

11. $f(x) = \log_3(x - 1)$

12. $g(x) = \log(-x)$

13. $f(x) = 2 - \log_2 x$

14. $f(x) = 3 + \log_5(x + 4)$

15. $g(x) = 2 \ln x$

16. $g(x) = \ln(x^2)$

17–20 ■ Domain Find the domain of the function.

17. $f(x) = 10^{x^2} + \log(1 - 2x)$

18. $g(x) = \log(2 + x - x^2)$

19. $h(x) = \ln(x^2 - 4)$

20. $k(x) = \ln|x|$

21–24 ■ Exponential Form Write the equation in exponential form.

21. $\log_2 1024 = 10$

22. $\log_6 37 = x$

23. $\log x = y$

24. $\ln c = 17$

25–28 ■ Logarithmic Form Write the equation in logarithmic form.

25. $2^6 = 64$

26. $49^{-1/2} = \frac{1}{7}$

27. $10^x = 74$

28. $e^k = m$

29–44 ■ Evaluating Logarithmic Expressions Evaluate the expression without using a calculator.

29. $\log_2 128$

30. $\log_8 1$

31. $10^{\log 45}$

32. $\log 0.000001$

33. $\ln(e^6)$

34. $\log_4 8$

35. $\log_3\left(\frac{1}{27}\right)$

36. $2^{\log_2 13}$

37. $\log_5 \sqrt{5}$

38. $e^{2 \ln 7}$

39. $\log 25 + \log 4$

40. $\log_3 \sqrt{243}$

41. $\log_2 16^{23}$

42. $\log_5 250 - \log_5 2$

43. $\log_8 6 - \log_8 3 + \log_8 2$

44. $\log \log 10^{100}$

45–50 ■ Expanding Logarithmic Expressions Expand the logarithmic expression.

45. $\log(AB^2C^3)$

46. $\log_2(x\sqrt{x^2 + 1})$

47. $\ln\sqrt{\dfrac{x^2 - 1}{x^2 + 1}}$

48. $\log\left(\dfrac{4x^3}{y^2(x - 1)^5}\right)$

49. $\log_5\left(\dfrac{x^2(1 - 5x)^{3/2}}{\sqrt{x^3 - x}}\right)$

50. $\ln\left(\dfrac{\sqrt[3]{x^4 + 12}}{(x + 16)\sqrt{x - 3}}\right)$

51–56 ■ Combining Logarithmic Expressions Combine into a single logarithm.

51. $\log 6 + 4 \log 2$

52. $\log x + \log(x^2y) + 3 \log y$

53. $\frac{3}{2} \log_2(x - y) - 2 \log_2(x^2 + y^2)$

54. $\log_5 2 + \log_5(x + 1) - \frac{1}{3} \log_5(3x + 7)$

55. $\log(x - 2) + \log(x + 2) - \frac{1}{2} \log(x^2 + 4)$

56. $\frac{1}{2}[\ln(x - 4) + 5 \ln(x^2 + 4x)]$

57–70 ■ Exponential and Logarithmic Equations Solve the equation. Find the exact solution if possible; otherwise, use a calculator to approximate to two decimals.

57. $3^{2x-7} = 27$

58. $5^{4-x} = \frac{1}{125}$

59. $2^{3x-5} = 7$

60. $10^{6-3x} = 18$

61. $4^{1-x} = 3^{2x+5}$

62. $e^{3x/4} = 10$

63. $x^2e^{2x} + 2xe^{2x} = 8e^{2x}$

64. $3^{2x} - 3^x - 6 = 0$

65. $\log x + \log(x + 1) = \log 12$

66. $\ln(x - 2) + \ln 3 = \ln(5x - 7)$

67. $\log_2(1 - x) = 4$

68. $\ln(2x - 3) + 1 = 0$

69. $\log_3(x - 8) + \log_3 x = 2$

70. $\log_8(x + 5) - \log_8(x - 2) = 1$

71–74 ■ Exponential Equations Use a calculator to find the solution of the equation, rounded to six decimal places.

71. $5^{-2x/3} = 0.63$

72. $2^{3x-5} = 7$

73. $5^{2x+1} = 3^{4x-1}$

74. $e^{-15k} = 10{,}000$

75–78 ■ Local Extrema and Asymptotes Draw a graph of the function and use it to determine the asymptotes and the local maximum and minimum values.

75. $y = e^{x/(x+2)}$

76. $y = 10^x - 5^x$

77. $y = \log(x^3 - x)$

78. $y = 2x^2 - \ln x$

79–80 ■ Solving Equations Find the solutions of the equation, rounded to two decimal places.

79. $3 \log x = 6 - 2x$

80. $4 - x^2 = e^{-2x}$

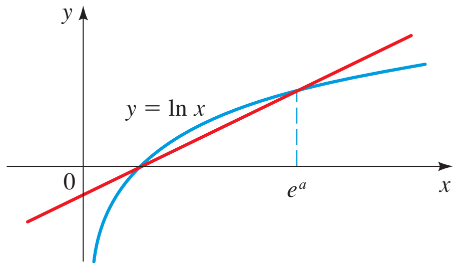
81–82 ■ Solving Inequalities Solve the inequality graphically.

81. $\ln x > x - 2$

82. $e^x < 4x^2$

83. Increasing and Decreasing Use a graph of $f(x) = e^x - 3e^{-x} - 4x$ to find, approximately, the intervals on which f is increasing and on which f is decreasing.

84. Equation of a Line Find an equation of the line shown in the figure.



85–88 ■ Change of Base Use the Change of Base Formula to evaluate the logarithm, rounded to six decimal places.

85. $\log_4 15$

86. $\log_7\left(\frac{3}{4}\right)$

87. $\log_9 0.28$

88. $\log_{100} 250$

89. Comparing Logarithms Which is larger, $\log_4 258$ or $\log_5 620$?

90. Inverse Function Find the inverse of the function $f(x) = 2^{3^x}$, and state its domain and range.

91. Compound Interest If $12,000 is invested at an interest rate of 10% per year, find the amount of the investment at the end of 3 years for each compounding method.

(a) Semiannually **(b)** Monthly

(c) Daily **(d)** Continuously

92. Compound Interest A sum of $5000 is invested at an interest rate of $8\frac{1}{2}$% per year, compounded semiannually.

(a) Find the amount of the investment after $1\frac{1}{2}$ years.

(b) After what period of time will the investment amount to $7000?

(c) If interest were compounded continously instead of semiannually, how long would it take for the amount to grow to $7000?

93. Compound Interest A money market account pays 5.2% annual interest, compounded daily. If $100,000 is invested in this account, how long will it take for the account to accumulate $10,000 in interest?

94. Compound Interest A retirement savings plan pays 4.5% interest, compounded continuously. How long will it take for an investment in this plan to double?

95–96 ■ APY Determine the annual percentage yield (APY) for the given nominal annual interest rate and compounding frequency.

95. 4.25%; daily

96. 3.2%; monthly

97. Cat Population The stray-cat population in a small town grows exponentially. In 1999 the town had 30 stray cats, and the relative growth rate was 15% per year.

(a) Find a function that models the stray-cat population $n(t)$ after t years.

(b) Find the projected population after 4 years.

(c) Find the number of years required for the stray-cat population to reach 500.

98. Bacterial Growth A culture contains 10,000 bacteria initially. After 1 hour the bacteria count is 25,000.

(a) Find the doubling period.

(b) Find the number of bacteria after 3 hours.

99. Radioactive Decay Uranium-234 has a half-life of 2.7×10^5 years.

(a) Find the amount remaining from a 10-mg sample after a thousand years.

(b) How long will it take this sample to decompose until its mass is 7 mg?

100. Radioactive Decay A sample of bismuth-210 decayed to 33% of its original mass after 8 days.

(a) Find the half-life of this element.

(b) Find the mass remaining after 12 days.

101. Radioactive Decay The half-life of radium-226 is 1590 years.

(a) If a sample has a mass of 150 mg, find a function that models the mass that remains after t years.

(b) Find the mass that will remain after 1000 years.

(c) After how many years will only 50 mg remain?

102. Radioactive Decay The half-life of palladium-100 is 4 days. After 20 days a sample has been reduced to a mass of 0.375 g.

(a) What was the initial mass of the sample?

(b) Find a function that models the mass remaining after t days.

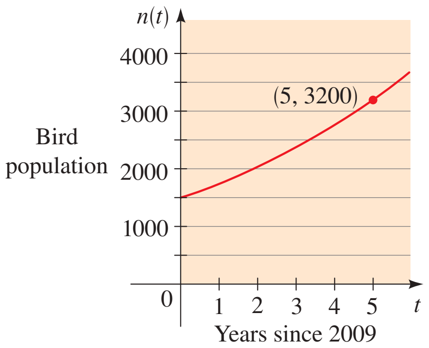
(c) What is the mass after 3 days?

(d) After how many days will only 0.15 g remain?

103. Bird Population The graph shows the population of a rare species of bird, where t represents years since 2009 and $n(t)$ is measured in thousands.

(a) Find a function that models the bird population at time t in the form $n(t) = n_0 e^{rt}$.

(b) What is the bird population expected to be in the year 2020?



104. Law of Cooling A car engine runs at a temperature of 190°F. When the engine is turned off, it cools according to Newton's Law of Cooling with constant $k = 0.0341$, where the time is measured in minutes. Find the time needed for the engine to cool to 90°F if the surrounding temperature is 60°F.

105. pH The hydrogen ion concentration of fresh egg whites was measured as

$$[H^+] = 1.3 \times 10^{-8} \text{ M}$$

Find the pH, and classify the substance as acidic or basic.

106. pH The pH of lime juice is 1.9. Find the hydrogen ion concentration.

107. Richter Scale If one earthquake has magnitude 6.5 on the Richter scale, what is the magnitude of another quake that is 35 times as intense?

108. Decibel Scale The drilling of a jackhammer was measured at 132 dB. The sound of whispering was measured at 28 dB. Find the ratio of the intensity of the drilling to that of the whispering.

CHAPTER 4 TEST

1. Sketch the graph of each function, and state its domain, range, and asymptote. Show the x- and y-intercepts on the graph.

(a) $f(x) = 2^{-x} + 4$ **(b)** $g(x) = \log_3(x + 3)$

2. Find the domain of the function.

(a) $f(t) = \ln(2t - 3)$ **(b)** $g(x) = \log(x^2 - 1)$

3. **(a)** Write the equation $6^{2x} = 25$ in logarithmic form.

(b) Write the equation $\ln A = 3$ in exponential form.

4. Find the exact value of the expression.

(a) $10^{\log 36}$ **(b)** $\ln e^3$ **(c)** $\log_3 \sqrt{27}$

(d) $\log_2 80 - \log_2 10$ **(e)** $\log_8 4$ **(f)** $\log_6 4 + \log_6 9$

5. Use the Laws of Logarithms to expand the expression.

(a) $\log\left(\dfrac{xy^3}{z^2}\right)$ **(b)** $\ln \sqrt{\dfrac{x}{y}}$ **(c)** $\log \sqrt[3]{\dfrac{x + 2}{x^4(x^2 + 4)}}$

6. Use the Laws of Logarithms to combine the expression into a single logarithm.

(a) $\log a + 2 \log b$ **(b)** $\ln(x^2 - 25) - \ln(x + 5)$ **(c)** $\log_2 3 - 3 \log_2 x + \frac{1}{2} \log_2(x + 1)$

7. Find the solution of the exponential equation, rounded to two decimal places.

(a) $3^{4x} = 3^{100}$ **(b)** $e^{3x-2} = e^{x^2}$ **(c)** $5^{x/10} + 1 = 7$ **(d)** $10^{x+3} = 6^{2x}$

8. Solve the logarithmic equation for x.

(a) $\log(2x) = 3$ **(b)** $\log(x + 1) + \log 2 = \log(5x)$

(c) $5 \ln(3 - x) = 4$ **(d)** $\log_2(x + 2) + \log_2(x - 1) = 2$

9. Use the Change of Base Formula to evaluate $\log_{12} 27$.

10. The initial size of a culture of bacteria is 1000. After 1 hour the bacteria count is 8000.

(a) Find a function $n(t) = n_0 e^{rt}$ that models the population after t hours.

(b) Find the population after 1.5 hours.

(c) After how many hours will the number of bacteria reach 15,000?

(d) Sketch the graph of the population function.

11. Suppose that \$12,000 is invested in a savings account paying 5.6% interest per year.

(a) Write the formula for the amount in the account after t years if interest is compounded monthly.

(b) Find the amount in the account after 3 years if interest is compounded daily.

(c) How long will it take for the amount in the account to grow to \$20,000 if interest is compounded continuously?

12. The half-life of krypton-91 (^{91}Kr) is 10 s. At time $t = 0$ a heavy canister contains 3 g of this radioactive gas.

(a) Find a function $m(t) = m_0 2^{-t/h}$ that models the amount of ^{91}Kr remaining in the canister after t seconds.

(b) Find a function $m(t) = m_0 e^{-rt}$ that models the amount of ^{91}Kr remaining in the canister after t seconds.

(c) How much ^{91}Kr remains after 1 min?

(d) After how long will the amount of ^{91}Kr remaining be reduced to 1 μg (1 microgram, or 10^{-6} g)?

13. An earthquake measuring 6.4 on the Richter scale struck Japan in July 2007, causing extensive damage. Earlier that year, a minor earthquake measuring 3.1 on the Richter scale was felt in parts of Pennsylvania. How many times more intense was the Japanese earthquake than the Pennsylvania earthquake?

A CUMULATIVE REVIEW TEST FOR CHAPTERS 2, 3, AND 4 CAN BE FOUND AT THE BOOK COMPANION WEBSITE: **www.stewartmath.com**.

Fitting Exponential and Power Curves to Data

In a previous *Focus on Modeling* (page 325) we learned that the shape of a scatter plot helps us to choose the type of curve to use in modeling data. The first plot in Figure 1 strongly suggests that a line be fitted through it, and the second one points to a cubic polynomial. For the third plot it is tempting to fit a second-degree polynomial. But what if an exponential curve fits better? How do we decide this? In this section we learn how to fit exponential and power curves to data and how to decide which type of curve fits the data better. We also learn that for scatter plots like those in the last two plots in Figure 1, the data can be modeled by logarithmic or logistic functions.

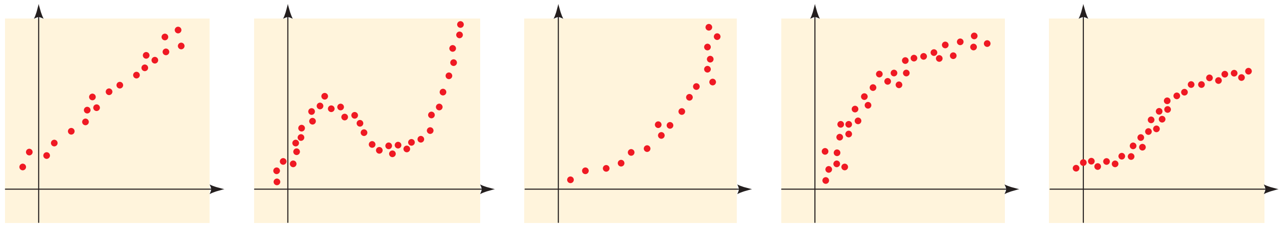


FIGURE 1

Modeling with Exponential Functions

If a scatter plot shows that the data increase rapidly, we might want to model the data using an *exponential model*, that is, a function of the form

$$f(x) = Ce^{kx}$$

where C and k are constants. In the first example we model world population by an exponential model. Recall from Section 4.6 that population tends to increase exponentially.

EXAMPLE 1 ■ An Exponential Model for World Population

Table 1 gives the population of the world in the 20th century.

(a) Draw a scatter plot, and note that a linear model is not appropriate.

(b) Find an exponential function that models population growth.

(c) Draw a graph of the function that you found together with the scatter plot. How well does the model fit the data?

(d) Use the model that you found to predict world population in the year 2020.

TABLE 1
World population

Year (t)	World population (P in millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2520
1960	3020
1970	3700
1980	4450
1990	5300
2000	6060

SOLUTION

(a) The scatter plot is shown in Figure 2. The plotted points do not appear to lie along a straight line, so a linear model is not appropriate.

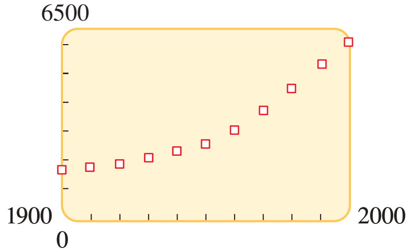


FIGURE 2 Scatter plot of world population

The population of the world increases exponentially.

(b) Using a graphing calculator and the `ExpReg` command (see Figure 3(a)), we get the exponential model

$$P(t) = (0.0082543) \cdot (1.0137186)^t$$

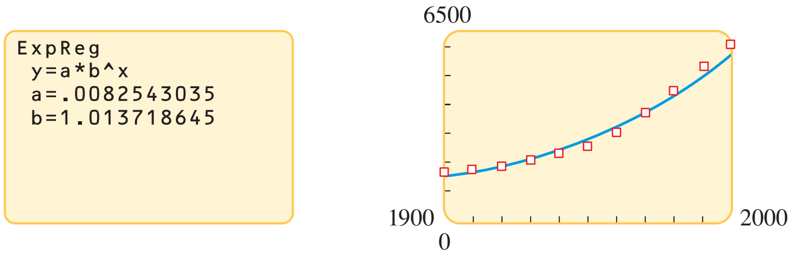
This is a model of the form $y = Cb^t$. To convert this to the form $y = Ce^{kt}$, we use the properties of exponentials and logarithms as follows.

$$\begin{aligned} 1.0137186^t &= e^{\ln 1.0137186^t} && A = e^{\ln A} \\ &= e^{t \ln 1.0137186} && \ln A^B = B \ln A \\ &= e^{0.013625t} && \ln 1.0137186 \approx 0.013625 \end{aligned}$$

Thus we can write the model as

$$P(t) = 0.0082543e^{0.013625t}$$

(c) From the graph in Figure 3(b) we see that the model appears to fit the data fairly well. The period of relatively slow population growth is explained by the depression of the 1930s and the two world wars.



(a) (b)

FIGURE 3 Exponential model for world population

(d) The model predicts that the world population in 2020 will be

$$\begin{aligned} P(2020) &= 0.0082543e^{(0.013625)(2020)} \\ &\approx 7{,}405{,}400{,}000 \end{aligned}$$

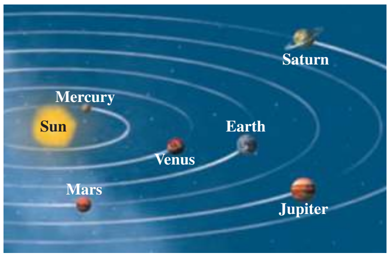
Modeling with Power Functions

If the scatter plot of the data we are studying resembles the graph of $y = ax^2$, $y = ax^{1.32}$, or some other power function, then we seek a *power model*, that is, a function of the form

$$f(x) = ax^n$$

where a is a positive constant and n is any real number.

In the next example we seek a power model for some astronomical data. In astronomy, distance in the solar system is often measured in astronomical units. An *astronomical unit* (AU) is the mean distance from the earth to the sun. The *period* of a planet is the time it takes the planet to make a complete revolution around the sun (measured in earth years). In this example we derive the remarkable relationship, first discovered by Johannes Kepler (see page 808), between the mean distance of a planet from the sun and its period.



EXAMPLE 2 ■ A Power Model for Planetary Periods

Table 2 gives the mean distance d of each planet from the sun in astronomical units and its period T in years.

TABLE 2
Distances and periods of the planets

Planet	d	T
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.523	1.881
Jupiter	5.203	11.861
Saturn	9.541	29.457
Uranus	19.190	84.008
Neptune	30.086	164.784
Pluto*	39.507	248.350

*Pluto is a "dwarf planet."

(a) Sketch a scatter plot. Is a linear model appropriate?
(b) Find a power function that models the data.
(c) Draw a graph of the function you found and the scatter plot on the same graph. How well does the model fit the data?
(d) Use the model that you found to calculate the period of an asteroid whose mean distance from the sun is 5 AU.

SOLUTION

(a) The scatter plot shown in Figure 4 indicates that the plotted points do not lie along a straight line, so a linear model is not appropriate.

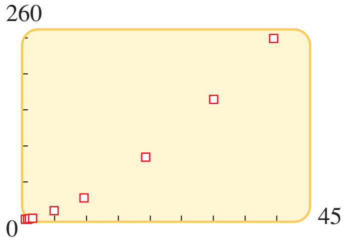


FIGURE 4 Scatter plot of planetary data

(b) Using a graphing calculator and the `PwrReg` command (see Figure 5(a)), we get the power model

$$T = 1.000396d^{1.49966}$$

If we round both the coefficient and the exponent to three significant figures, we can write the model as

$$T = d^{1.5}$$

This is the relationship discovered by Kepler (see page 808). Sir Isaac Newton (page 911) later used his Law of Gravity to derive this relationship theoretically, thereby providing strong scientific evidence that the Law of Gravity must be true.

(c) The graph is shown in Figure 5(b). The model appears to fit the data very well.

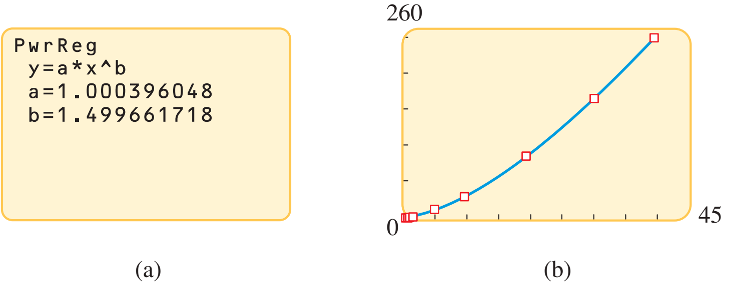


FIGURE 5 Power model for planetary data

(d) In this case $d = 5$ AU, so our model gives

$$T = 1.00039 \cdot 5^{1.49966} \approx 11.22$$

The period of the asteroid is about 11.2 years. ■

Linearizing Data

We have used the shape of a scatter plot to decide which type of model to use: linear, exponential, or power. This works well if the data points lie on a straight line. But it's difficult to distinguish a scatter plot that is exponential from one that requires a power model. So to help decide which model to use, we can *linearize* the data, that is, apply a function that "straightens" the scatter plot. The inverse of the linearizing function is then an appropriate model. We now describe how to linearize data that can be modeled by exponential or power functions.

Linearizing Exponential Data

If we suspect that the data points (x, y) lie on an exponential curve $y = Ce^{kx}$, then the points

$$(x, \ln y)$$

should lie on a straight line. We can see this from the following calculations.

$$\begin{aligned} \ln y &= \ln Ce^{kx} && \text{Assume that } y = Ce^{kx} \text{ and take ln} \\ &= \ln e^{kx} + \ln C && \text{Property of ln} \\ &= kx + \ln C && \text{Property of ln} \end{aligned}$$

To see that $\ln y$ is a linear function of x, let $Y = \ln y$ and $A = \ln C$; then

$$Y = kx + A$$

TABLE 3
World population data

t	Population P (in millions)	$\ln P$
1900	1650	21.224
1910	1750	21.283
1920	1860	21.344
1930	2070	21.451
1940	2300	21.556
1950	2520	21.648
1960	3020	21.829
1970	3700	22.032
1980	4450	22.216
1990	5300	22.391
2000	6060	22.525

We apply this technique to the world population data (t, P) to obtain the points $(t, \ln P)$ in Table 3. The scatter plot of $(t, \ln P)$ in Figure 6, called a **semi-log plot**, shows that the linearized data lie approximately on a straight line, so an exponential model should be appropriate.

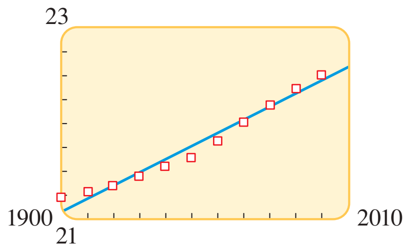


FIGURE 6 Semi-log plot of data in Table 3

Linearizing Power Data

If we suspect that the data points (x, y) lie on a power curve $y = ax^n$, then the points

$$(\ln x, \ln y)$$

should be on a straight line. We can see this from the following calculations.

$$\begin{aligned} \ln y &= \ln ax^n && \text{Assume that } y = ax^n \text{ and take ln} \\ &= \ln a + \ln x^n && \text{Property of ln} \\ &= \ln a + n \ln x && \text{Property of ln} \end{aligned}$$

To see that $\ln y$ is a linear function of $\ln x$, let $Y = \ln y$, $X = \ln x$, and $A = \ln a$; then

$$Y = nX + A$$

TABLE 4
Log-log table

$\ln d$	$\ln T$
−0.94933	−1.4230
−0.32435	−0.48613
0	0
0.42068	0.6318
1.6492	2.4733
2.2556	3.3829
2.9544	4.4309
3.4041	5.1046
3.6765	5.5148

We apply this technique to the planetary data (d, T) in Table 2 to obtain the points $(\ln d, \ln T)$ in Table 4. The scatter plot of $(\ln d, \ln T)$ in Figure 7, called a **log-log plot**, shows that the data lie on a straight line, so a power model seems appropriate.

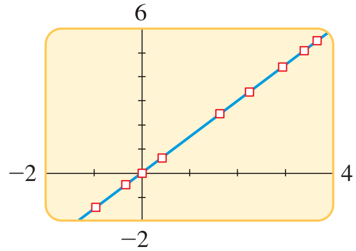


FIGURE 7 Log-log plot of data in Table 4

An Exponential or Power Model?

Suppose that a scatter plot of the data points (x, y) shows a rapid increase. Should we use an exponential function or a power function to model the data? To help us decide, we draw two scatter plots: one for the points $(x, \ln y)$ and the other for the points $(\ln x, \ln y)$. If the first scatter plot appears to lie along a line, then an exponential model is appropriate. If the second plot appears to lie along a line, then a power model is appropriate.

EXAMPLE 3 ■ An Exponential or Power Model?

Data points (x, y) are shown in Table 5.

(a) Draw a scatter plot of the data.

(b) Draw scatter plots of $(x, \ln y)$ and $(\ln x, \ln y)$.

(c) Is an exponential function or a power function appropriate for modeling this data?

(d) Find an appropriate function to model the data.

TABLE 5

x	y
1	2
2	6
3	14
4	22
5	34
6	46
7	64
8	80
9	102
10	130

SOLUTION

(a) The scatter plot of the data is shown in Figure 8.

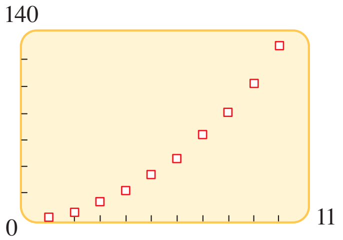


FIGURE 8

(b) We use the values from Table 6 to graph the scatter plots in Figures 9 and 10.

TABLE 6

x	$\ln x$	$\ln y$
1	0	0.7
2	0.7	1.8
3	1.1	2.6
4	1.4	3.1
5	1.6	3.5
6	1.8	3.8
7	1.9	4.2
8	2.1	4.4
9	2.2	4.6
10	2.3	4.9

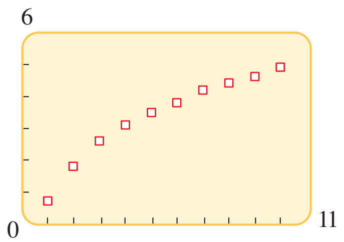


FIGURE 9 Semi-log plot

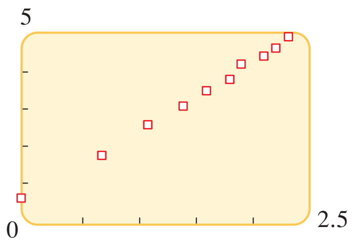


FIGURE 10 Log-log plot

(c) The scatter plot of $(x, \ln y)$ in Figure 9 does not appear to be linear, so an exponential model is not appropriate. On the other hand, the scatter plot of $(\ln x, \ln y)$ in Figure 10 is very nearly linear, so a power model is appropriate.

(d) Using the `PwrReg` command on a graphing calculator, we find that the power function that best fits the data point is

$$y = 1.85x^{1.82}$$

The graph of this function and the original data points are shown in Figure 11. ■

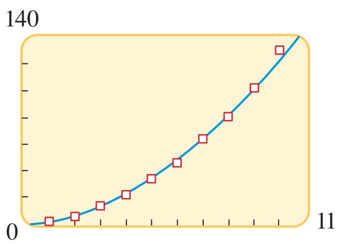


FIGURE 11

Before graphing calculators and statistical software became common, exponential and power models for data were often constructed by first finding a linear model for the linearized data. Then the model for the actual data was found by taking exponentials. For instance, if we find that $\ln y = A \ln x + B$, then by taking exponentials we get the model $y = e^B \cdot e^{A \ln x}$, or $y = Cx^A$ (where $C = e^B$). Special graphing paper called "log paper" or "log-log paper" was used to facilitate this process.

Modeling with Logistic Functions

A logistic growth model is a function of the form

$$f(t) = \frac{c}{1 + ae^{-bt}}$$

where a, b, and c are positive constants. Logistic functions are used to model populations where the growth is constrained by available resources. (See Exercises 27–30 of Section 4.2.)

EXAMPLE 4 ■ Stocking a Pond with Catfish

TABLE 7

Week	Catfish
0	1000
15	1500
30	3300
45	4400
60	6100
75	6900
90	7100
105	7800
120	7900

Much of the fish that is sold in supermarkets today is raised on commercial fish farms, not caught in the wild. A pond on one such farm is initially stocked with 1000 catfish, and the fish population is then sampled at 15-week intervals to estimate its size. The population data are given in Table 7.

(a) Find an appropriate model for the data.

(b) Make a scatter plot of the data and graph the model that you found in part (a) on the scatter plot.

(c) How does the model predict that the fish population will change with time?

SOLUTION

(a) Since the catfish population is restricted by its habitat (the pond), a logistic model is appropriate. Using the `Logistic` command on a calculator (see Figure 12(a)), we find the following model for the catfish population $P(t)$:

$$P(t) = \frac{7925}{1 + 7.7e^{-0.052t}}$$

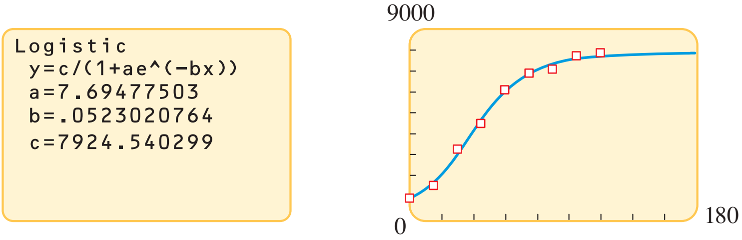


FIGURE 12 (a) (b) Catfish population $y = P(t)$

(b) The scatter plot and the logistic curve are shown in Figure 12(b).

(c) From the graph of P in Figure 12(b) we see that the catfish population increases rapidly until about $t = 80$ weeks. Then growth slows down, and at about $t = 120$ weeks the population levels off and remains more or less constant at slightly over 7900. ■

The behavior that is exhibited by the catfish population in Example 4 is typical of logistic growth. After a rapid growth phase, the population approaches a constant level called the **carrying capacity** of the environment. This occurs because as $t \to \infty$, we have $e^{-bt} \to 0$ (see Section 4.2), and so

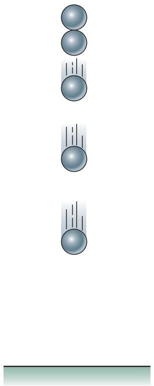
$$P(t) = \frac{c}{1 + ae^{-bt}} \longrightarrow \frac{c}{1 + 0} = c$$

Thus the carrying capacity is c.

PROBLEMS

1. **U.S. Population** The U.S. Constitution requires a census every 10 years. The census data for 1790–2010 are given in the table.
   (a) Make a scatter plot of the data.
   (b) Use a calculator to find an exponential model for the data.
   (c) Use your model to predict the population at the 2020 census.
   (d) Use your model to estimate the population in 1965.

Year	Population (in millions)	Year	Population (in millions)	Year	Population (in millions)
1790	3.9	1870	38.6	1950	151.3
1800	5.3	1880	50.2	1960	179.3
1810	7.2	1890	63.0	1970	203.3
1820	9.6	1900	76.2	1980	226.5
1830	12.9	1910	92.2	1990	248.7
1840	17.1	1920	106.0	2000	281.4
1850	23.2	1930	123.2	2010	308.7
1860	31.4	1940	132.2		

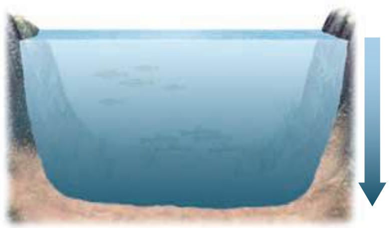


Time (s)	Distance (m)
0.1	0.048
0.2	0.197
0.3	0.441
0.4	0.882
0.5	1.227
0.6	1.765
0.7	2.401
0.8	3.136
0.9	3.969
1.0	4.902

2. **A Falling Ball** In a physics experiment a lead ball is dropped from a height of 5 m. The students record the distance the ball has fallen every one-tenth of a second. (This can be done by using a camera and a strobe light.) Their data are shown in the margin.
   (a) Make a scatter plot of the data.
   (b) Use a calculator to find a power model.
   (c) Use your model to predict how far a dropped ball would fall in 3 s.

3. **Half-Life of Radioactive Iodine** A student is trying to determine the half-life of radioactive iodine-131. He measures the amount of iodine-131 in a sample solution every 8 hours. His data are shown in the table below.
   (a) Make a scatter plot of the data.
   (b) Use a calculator to find an exponential model.
   (c) Use your model to find the half-life of iodine-131.

Time (h)	Amount of ^{131}I (g)
0	4.80
8	4.66
16	4.51
24	4.39
32	4.29
40	4.14
48	4.04



Light intensity decreases exponentially with depth.

4. **The Beer-Lambert Law** As sunlight passes through the waters of lakes and oceans, the light is absorbed, and the deeper it penetrates, the more its intensity diminishes. The light intensity I at depth x is given by the Beer-Lambert Law:

$$I = I_0 e^{-kx}$$

where I_0 is the light intensity at the surface and k is a constant that depends on the murkiness of the water (see page 366). A biologist uses a photometer to investigate light penetration in a northern lake, obtaining the data in the table.

(a) Use a graphing calculator to find an exponential function of the form given by the Beer-Lambert Law to model these data. What is the light intensity I_0 at the surface on this day, and what is the "murkiness" constant k for this lake? [*Hint:* If your calculator gives you a function of the form $I = ab^x$, convert this to the form you want using the identities $b^x = e^{\ln(b^x)} = e^{x \ln b}$. See Example 1(b).]

(b) Make a scatter plot of the data, and graph the function that you found in part (a) on your scatter plot.

(c) If the light intensity drops below 0.15 lumen (lm), a certain species of algae can't survive because photosynthesis is impossible. Use your model from part (a) to determine the depth below which there is insufficient light to support this algae.

Depth (ft)	Light intensity (lm)	Depth (ft)	Light intensity (lm)
5	13.0	25	1.8
10	7.6	30	1.1
15	4.5	35	0.5
20	2.7	40	0.3

Time	Words recalled
15 min	64.3
1 h	45.1
8 h	37.3
1 day	32.8
2 days	26.9
3 days	25.6
5 days	22.9

5. Experimenting with "Forgetting" Curves Every one of us is all too familiar with the phenomenon of forgetting. Facts that we clearly understood at the time we first learned them sometimes fade from our memory by the time the final exam rolls around. Psychologists have proposed several ways to model this process. One such model is Ebbinghaus' Law of Forgetting, described on page 356. Other models use exponential or logarithmic functions. To develop her own model, a psychologist performs an experiment on a group of volunteers by asking them to memorize a list of 100 related words. She then tests how many of these words they can recall after various periods of time. The average results for the group are shown in the table.

(a) Use a graphing calculator to find a *power* function of the form $y = at^b$ that models the average number of words y that the volunteers remember after t hours. Then find an *exponential* function of the form $y = ab^t$ to model the data.

(b) Make a scatter plot of the data, and graph both the functions that you found in part (a) on your scatter plot.

(c) Which of the two functions seems to provide the better model?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The number of different bat species in a cave is related to the size of the cave by a power function.

6. Modeling the Species-Area Relation The table gives the areas of several caves in central Mexico and the number of bat species that live in each cave.*

(a) Find a power function that models the data.

(b) Draw a graph of the function you found in part (a) and a scatter plot of the data on the same graph. Does the model fit the data well?

(c) The cave called El Sapo near Puebla, Mexico, has a surface area of $A = 205\ \text{m}^2$. Use the model to estimate the number of bat species you would expect to find in that cave.

Cave	Area (m^2)	Number of species
La Escondida	18	1
El Escorpion	19	1
El Tigre	58	1
Mision Imposible	60	2
San Martin	128	5
El Arenal	187	4
La Ciudad	344	6
Virgen	511	7

*A. K. Brunet and R. A. Medallin, "The Species-Area Relationship in Bat Assemblages of Tropical Caves." *Journal of Mammalogy, 82*(4):1114–1122, 2001.

7. Auto Exhaust Emissions A study by the U.S. Office of Science and Technology in 1972 estimated the cost of reducing automobile emissions by certain percentages. Find an exponential model that captures the "diminishing returns" trend of these data shown in the table below.

Reduction in emissions (%)	Cost per car ($)
50	45
55	55
60	62
65	70
70	80
75	90
80	100
85	200
90	375
95	600

8. Exponential or Power Model? Data points (x, y) are shown in the table.

(a) Draw a scatter plot of the data.

(b) Draw scatter plots of $(x, \ln y)$ and $(\ln x, \ln y)$.

(c) Which is more appropriate for modeling this data: an exponential function or a power function?

(d) Find an appropriate function to model the data.

x	2	4	6	8	10	12	14	16
y	0.08	0.12	0.18	0.25	0.36	0.52	0.73	1.06

9. Exponential or Power Model? Data points (x, y) are shown in the table in the margin.

(a) Draw a scatter plot of the data.

(b) Draw scatter plots of $(x, \ln y)$ and $(\ln x, \ln y)$.

(c) Which is more appropriate for modeling this data: an exponential function or a power function?

(d) Find an appropriate function to model the data.

x	10	20	30	40	50	60	70	80	90
y	29	82	151	235	330	430	546	669	797

10. Logistic Population Growth The table and scatter plot give the population of black flies in a closed laboratory container over an 18-day period.

(a) Use the `Logistic` command on your calculator to find a logistic model for these data.

(b) Use the model to estimate the time when there were 400 flies in the container.

Time (days)	Number of flies
0	10
2	25
4	66
6	144
8	262
10	374
12	446
16	492
18	498