September 6th

What is a fraction? I don't know, some kind of number. What can we *do* with fractions? We can solve problems like 'Find a number x which makes the equation

$$3x + 5 = 7$$

true."

Here, subtract 5 from both sides to get the equation

$$3x = 2$$

and then divide both sides by 3 - which is the same as multiplying both sides by the fraction $\frac{1}{3}$ - to get

$$x = \frac{2}{3}$$

This works for any **linear equation**:

$$\begin{aligned} 7x + 12 &= 3 \\ 9x - 4 &= 5 \\ (-13) \cdot x + 8 &= 1 \end{aligned}$$

(here I am using **coefficients** which are integers, but you can also solve any linear equation with fractions as coefficients, and get a fraction answer. I will say what 'linear' and 'coefficient' mean later.)

Another way of thinking about the fraction $\frac{2}{3}$ is 'the number that multiplies 3 to get 2', or in an equation 'the number x that makes

$$3x = 2$$

true'.

Fact 1. *If $\frac{a}{k}$ and $\frac{b}{k}$ are two fractions with the same denominator, then*

$$\frac{a}{k} + \frac{b}{k} = \frac{a+b}{k}$$

Reason. Take the left side and multiply by k:

$$k \cdot \left(\frac{a}{k} + \frac{b}{k} \right)$$

The distributive law says

$$k \cdot \left(\frac{a}{k} + \frac{b}{k} \right) = k \cdot \frac{a}{k} + k \cdot \frac{b}{k}$$

Since we said $\frac{a}{k}$ is exactly the number we multiply by k to get a, this simplifies

$$\begin{aligned} k \cdot \left(\frac{a}{k} + \frac{b}{k} \right) &= k \cdot \frac{a}{k} + k \cdot \frac{b}{k} \\ &= a + b \end{aligned}$$

This says $\frac{a}{k} + \frac{b}{k}$ is the number that you multiply by k to get $a + b$, which we already said is what $\frac{a+b}{k}$ means. So

$$\frac{a}{k} + \frac{b}{k} = \frac{a+b}{k}$$

□

So, if we want to add two fractions, it is easiest when they have the same denominator. We know that the same fraction can be written in different ways (e.g., $\frac{8}{12} = \frac{10}{15} = \frac{2}{3}$), so we rewrite the fractions to have the same denominator, and then add like above.

Some examples:

$$\begin{aligned} \frac{2}{7} + \frac{5}{9} &= \frac{2}{7} \cdot 1 + 1 \cdot \frac{5}{9} \\ &= \frac{2}{7} \cdot \frac{9}{9} + \frac{7}{7} \cdot \frac{5}{9} \\ &= \frac{2 \cdot 9}{7 \cdot 9} + \frac{7 \cdot 5}{7 \cdot 9} \\ &= \frac{18}{63} + \frac{35}{63} \\ &= \frac{18 + 35}{63} \\ &= \frac{53}{63} \end{aligned}$$

When I multiply a number by 1, it doesn't change the number. If I think about that 1 as $\frac{9}{9}$, then fraction multiplication means I multiply the top and bottom by 9. If I do this for the second fraction, but use $\frac{7}{7}$ as my version of 1, the same thing happens. Then both fractions have the same denominator, $7 \cdot 9 = 63$.

Similarly:

$$\begin{aligned}
\frac{7}{3} + \frac{2}{5} &= \frac{7}{3} \cdot 1 + 1 \cdot \frac{2}{5} \\
&= \frac{7}{3} \cdot \frac{5}{5} + \frac{3}{3} \cdot \frac{2}{5} \\
&= \frac{7 \cdot 5}{3 \cdot 5} + \frac{3 \cdot 2}{3 \cdot 5} \\
&= \frac{35}{15} + \frac{6}{15} \\
&= \frac{35 + 6}{15} \\
&= \frac{41}{15}
\end{aligned}$$

Another example:

$$\begin{aligned}
\frac{5}{12} + \frac{5}{18} &= \frac{5}{12} \cdot 1 + 1 \cdot \frac{5}{18} \\
&= \frac{5}{12} \cdot \frac{18}{18} + \frac{12}{12} \cdot \frac{5}{18} \\
&= \frac{5 \cdot 18}{12 \cdot 18} + \frac{12 \cdot 5}{12 \cdot 18} \\
&= \frac{90}{216} + \frac{60}{216} \\
&= \frac{90 + 60}{216} \\
&= \frac{150}{216} = \frac{75 \cdot 2}{108 \cdot 2} \\
&= \frac{75}{108} = \frac{25 \cdot 3}{36 \cdot 3} \\
&= \frac{25}{36}
\end{aligned}$$

In this final example, we simplified the fraction we got at the end. If you spend a while thinking about this, you might notice that

$$36 = 12 \cdot 3$$

and

$$36 = 18 \cdot 2,$$

and then get the idea 'instead of multiplying by $\frac{18}{18}$ and $\frac{12}{12}$, to get a common denominator of 216, can we try for a common denominator of 36?'

Let's go for it:

$$\begin{aligned}
\frac{5}{12} + \frac{5}{18} &= \frac{5}{12} \cdot 1 + 1 \cdot \frac{5}{18} \\
&= \frac{5}{12} \cdot \frac{3}{3} + \frac{2}{2} \cdot \frac{5}{18} \\
&= \frac{5 \cdot 3}{12 \cdot 3} + \frac{2 \cdot 5}{2 \cdot 18} \\
&= \frac{15}{36} + \frac{10}{36} \\
&= \frac{15 + 10}{36} \\
&= \frac{25}{36}
\end{aligned}$$

That's a bit simpler, not so many steps, the multiplications are easier, etc.. Unfortunately, we had to notice that both 12 and 18 can divide 36 evenly.

Definition 1. *The **least common multiple** of two integers a and b, which we write as $lcm(a, b)$ is the smallest number that can be evenly divided by both a and b.*

Notice that there is always a number that can be evenly divided by a and b, since their product $a \cdot b$ is an example. Sometimes though, like $lcm(12, 18) = 36$, the least common multiple of two numbers is less than their product.

We can use this to try to make fraction addition easier: if we find the least common multiple of the denominators in the fraction, we can rewrite both fractions with that lcm as the common denominator.

Let me ignore this for the moment.

If we consider a general fraction addition problem, using the symbols a, b, c, and d to mean some integer numbers (with $b, d \neq 0$), we can repeat the above process:

$$\begin{aligned}
\frac{a}{b} + \frac{c}{d} &= \frac{a}{b} \cdot 1 + 1 \cdot \frac{c}{d} \\
&= \frac{a}{b} \cdot \frac{d}{d} + \frac{b}{b} \cdot \frac{c}{d} \\
&= \frac{a \cdot d}{b \cdot d} + \frac{b \cdot c}{b \cdot d} \\
&= \frac{a \cdot d}{b \cdot d} + \frac{b \cdot c}{b \cdot d} \\
&= \frac{ad + bc}{bd}
\end{aligned}$$

This is the usual formula for fraction addition. Notice this does not come from nowhere, it comes from repeating the process we used above to add fractions. Instead of memorizing the formula, I strongly encourage everyone to work through the process for many examples, writing cleanly as you do. This will build up your skills in a way that keeps, instead of memorizing a formula for a test and forgetting right after.

Sometimes, if your professor is really mean, you might see fractions within fractions. Here, one should work from the inside out, and do the numerator and denominator separately.

An example:

$$\begin{aligned}
\frac{\frac{1}{6}+\frac{3}{7}}{\frac{2}{3}+\frac{1}{5}} &= \frac{\frac{1}{6}\cdot 1+1\cdot\frac{3}{7}}{\frac{2}{3}+\frac{1}{5}}\\
&= \frac{\frac{1}{6}\cdot\frac{7}{7}+\frac{6}{6}\cdot\frac{3}{7}}{\frac{2}{3}+\frac{1}{5}}\\
&= \frac{\frac{1\cdot 7}{6\cdot 7}+\frac{6\cdot 3}{6\cdot 7}}{\frac{2}{3}+\frac{1}{5}}\\
&= \frac{\frac{7}{42}+\frac{18}{42}}{\frac{2}{3}+\frac{1}{5}}\\
&= \frac{\frac{25}{42}}{\frac{2}{3}+\frac{1}{5}}\\
&= \frac{\frac{25}{42}}{\frac{2}{3}\cdot 1+1\cdot\frac{1}{5}}\\
&= \frac{\frac{25}{42}}{\frac{2}{3}\cdot\frac{5}{5}+\frac{3}{3}\cdot\frac{1}{5}}\\
&= \frac{\frac{25}{42}}{\frac{2\cdot 5}{3\cdot 5}+\frac{3\cdot 1}{3\cdot 5}}\\
&= \frac{\frac{25}{42}}{\frac{10}{15}+\frac{3}{15}}\\
&= \frac{\frac{25}{42}}{\frac{13}{15}}\\
&= \frac{25}{42}\cdot\frac{15}{13}\\
&= \frac{375}{546}=\frac{125\cdot 3}{182\cdot 3}\\
&= \frac{125}{182}
\end{aligned}$$

Everything should make sense up to the fourth line from the end, with

$$\frac{\frac{25}{42}}{\frac{13}{15}}$$

Then I used the fact that dividing by a fraction is the same as multiplying by the **reciprocal** of that fraction, i.e. flip the fraction over. We haven't talked about this yet, but think about how dividing by 3 (or $\frac{3}{1}$) is the same as multiplying by $\frac{1}{3}$. Then there's the simplification step, which we still haven't said much about yet.

The point for this example is that we did the fraction addition 'inside' the top without doing anything to the bottom, and then once that was done we did the fraction addition 'inside' the bottom without affecting the top. Once it's all down to fractions and multiplication/division, life is (supposed to be) easy. Fractions like multiplications. Fractions were introduced to solve multiplication problems, so that makes sense. This is why fraction multiplication is easy, just multiply the tops and bottoms:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

Compound example.