September 18

Last time we talked about polynomial expressions.

Here, an **expression** is something written down using 0 or more variables and arithmetic or algebraic operations like $+$, $\times$, $-$, $\div$, radicals $\sqrt[n]{}$, and powers $*^n$ (or, when we get to trigonometry, sin and cos and so on).

The point is really that an expression does *not* involve an equals sign, or an inequality like less than $<$ or greater than $>$. An expression is just a single thing.

An **equation**, on the other hand, is a pair of expressions connected by an equality sign '='.

Similarly an **inequality** (for some reason we don't call it an 'inequation') is a pair of expressions connected by one of the inequality symbols, '$<$', '$>$', '$\leq$', or '$\geq$'.

For completeness I'll remind you that a **solution** or **root** of an equation with one variable, x, is a choice of number so that replacing x with that number makes both sides of the equation actually equal.

For instance, $x = 2$ is a solution to the equation

$$x^2 - 1 = x + 1$$

since replacing x with 2 turns both sides into the number 3.

(EXAMPLES)

Next, we discussed some useful equations: For any numbers a and b, we have

$$a^2 - b^2 = (a - b) \cdot (a + b)$$

This is not too hard to see: using the distributive law a bunch, we get

$$\begin{aligned}(a - b) \cdot (a + b) &= (a - b) \cdot a + (a - b) \cdot b \\ &= a \cdot a + (-b) \cdot a + a \cdot b + (-b) \cdot b \\ &= a^2 - ba + ab - b^2 \\ &= a^s - b^2\end{aligned}$$

This equation is known as the 'difference of squares', since it is about factoring a subtraction between two things which are squared.

This can be used to seem clever without a calculator: if I ask you to multiply 107 and 93, you might notice these are both 7 away from 100. So

$$\begin{aligned} 107 \cdot 93 &= (100+7) \cdot (100-7) \\ &= 100^2 - 7^2 \\ &= 10,000 - 49 \\ &= 9,951 \end{aligned}$$

Another example: consider 84 and 76. We see $84 = 80+4$ and $76 = 80-4$, so

$$\begin{aligned} 84 \cdot 76 &= (80+4) \cdot (80-4) \\ &= 80^2 - 4^2 \\ &= 6,400 - 16 \\ &= 6384 \end{aligned}$$

This works with any numbers, or even expressions involving variables like x or y if we wanted.

An example with radicals (going the other way, right-to-left in the formula above):

$$\begin{aligned} (\sqrt{5} - 2\sqrt{3}) \cdot (\sqrt{5} + 2\sqrt{3}) &= (\sqrt{5})^2 - (2\sqrt{3})^2 \\ &= (\sqrt{5})^2 - 2^2 \cdot \sqrt{3}^2 \\ &= 5 - 4 \cdot 3 \\ &= 5 - 12 \\ &= -7 \end{aligned}$$

Notice that when we square the radicals, we get integers. One place we often use the difference of squares is in 'rationalizing the denominator' of a fraction involving radicals. Since division is hard, and radicals are hard, we don't want to do two hard things at the same time. So we would like to rewrite an expression like

$$\frac{2+\sqrt{3}}{3+\sqrt{7}}$$

Taking inspiration from above, we multiply this fraction by a clever form of 1:

$$\begin{aligned}
\frac{2+\sqrt{3}}{3+\sqrt{7}} \cdot \frac{3-\sqrt{7}}{3-\sqrt{7}} &= \frac{(2+\sqrt{3})\cdot(3-\sqrt{7})}{(3+\sqrt{7})\cdot(3-\sqrt{7})} \\
&= \frac{(2+\sqrt{3})\cdot(3-\sqrt{7})}{3^2-(\sqrt{7})^2} \\
&= \frac{(2+\sqrt{3})\cdot(3-\sqrt{7})}{9-7} \\
&= \frac{(2+\sqrt{3})\cdot(3-\sqrt{7})}{2} \\
&= \frac{2\cdot 3+\sqrt{3}\cdot 3-2\sqrt{7}-\sqrt{3}\sqrt{7}}{2} \\
&= \frac{6+3\sqrt{3}-2\sqrt{7}-\sqrt{21}}{2}
\end{aligned}$$

Notice we chose $3-\sqrt{7}$ so that multiplying by $3+\sqrt{7}$ would get us into the difference of squares situation. This $3-\sqrt{7}$ is called the **conjugate** of $3+\sqrt{7}$. More generally, the conjugate of $a+b\sqrt{c}$ is $a-b\sqrt{c}$ for any numbers a, b, c, and is the number you multiply by to to get rid of the $\sqrt{}$

Another example: Simplify

$$\frac{4-\sqrt{2}}{2+\sqrt{2}}$$

Here, the conjugate of the denominator is $2-\sqrt{2}$, so we multiply top and bottom:

$$\begin{aligned}
\frac{4-\sqrt{2}}{2+\sqrt{2}} &= \frac{4-\sqrt{2}}{2+\sqrt{2}} \cdot \frac{2-\sqrt{2}}{2-\sqrt{2}} \\
&= \frac{(4-\sqrt{2})(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})} \\
&= \frac{(4-\sqrt{2})(2-\sqrt{2})}{2^2-(\sqrt{2})^2} \\
&= \frac{(4-\sqrt{2})(2-\sqrt{2})}{4-2} \\
&= \frac{(4-\sqrt{2})(2-\sqrt{2})}{2} \\
&= \frac{4\cdot 2 - 4\cdot\sqrt{2} - \sqrt{2}\cdot 2 + \sqrt{2}\sqrt{2}}{2} \\
&= \frac{8-4\sqrt{2}-2\sqrt{2}+2}{2} \\
&= \frac{10-6\sqrt{2}}{2} \\
&= 5-3\sqrt{2}
\end{aligned}$$

Here, we could check that this is actually the right answer, by checking that this answer times the denominator gives back the numerator:

$$\begin{aligned}
(5-3\sqrt{2})\cdot(2+\sqrt{2}) &= 10+5\sqrt{2}-3\sqrt{2}\cdot 2 - 3\sqrt{2}\cdot\sqrt{2} \\
&= 10+5\sqrt{2}-6\sqrt{2}-6 \\
&= 4-\sqrt{2}
\end{aligned}$$

which is the numerator of our original fraction.

Just like the difference of squares, there is a formula for the difference of cubes:

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

Again, this is not so bad to check. Expand the right-hand side:

$$\begin{aligned}
(a-b)(a^2+ab+b^2) &= (a-b)a^2+(a-b)ab+(a-b)b^2 \\
&= a\cdot a^2 - b\cdot a^2 + a\cdot ab - b\cdot ab + a\cdot b^2 - b\cdot b^2 \\
&= a^3-a^2b+a^2b-ab^2+ab^2-b^3 \\
&= a^3-b^3
\end{aligned}$$

There is also a difference of quartics (fourth power):

$$a^4 - b^4 = (a - b)(a^3 + a^2b + ab^2 + b^3)$$

and in fact a difference of n-th powers for any n.

In general (without expanding our number system) there is no nice factorization of $a^2 + b^2$, but it turns out there is a nice factorization of the sum of cubes:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

I'll let you check this by expanding the right-hand side.

We ended the day by talking about solving quadratic polynomials.

We already saw some nice cases where we can factor:

$$x^2 + 7x + 10 = 0$$

can be written as

$$(x + 2)(x + 5) = 0$$

and the only way for a product to be 0 is if one of the factors is 0. So either $x = -2$ or $x = -5$ will solve the equation.

To find the factors like this, remember that expanding gives

$$(x + a)(x + b) = x^2 + xb + ax + ab = x^2 + (a + b)x + ab$$

This means to factor

$$x^2 + px + q$$

we need to find numbers a and b so that $a+b = p$, and $ab = q$. For the above example, we want $a + b = 7$ and $ab = 10$, so we recognize $a = 2, b = 5$ works.

However, we cannot always factor in this way. My specific example was

$$x^2 + 17x - 23 = 0$$

You all memorized a 'quadratic formula' at some point, so you can write down the answer, but how would you think of it on your own? How would you come up with the quadratic equation?

Well, I lost track of time and didn't finish the discussion.