September 11

Do a compound example

Powers and Radicals: the 'n-th power' of a number x, written x^n, is the product of n copies of that number. For instance

$$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$$

or

$$7^3 = 7 \cdot 7 \cdot 7 = 49 \cdot 7 = 343$$

(since multiplication is associative, I don't need parentheses to say what I mean.) If we only have one copy of a number, we say the multiplication is just that number, so

$$x^1 = x$$

for any number x.

It is easy to see that taking powers follows the law

$$x^{n+m} = x^n \cdot x^m$$

since both sides are just $n + m$ many copies of x multiplied together.

For instance,

$$2^{3+2} = 2^3 \cdot 2^2 = (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2) = 8 \cdot 4 = 32$$

So taking powers 'turns addition into multiplication'. If we want to keep this rule, it makes sense to say the '0-th power' is always equal to 1:

$$x^0 = 1$$

since we have

$$x^n = x^{n+0} = x^n \cdot x^0,$$

so multiplication by x^0 should not change anything. (There is a slight confusion about what 0^0 should mean, since this equation is also true if $x^0 = 0$ for $x = 0$. In this class, we will always say $0^0 = 1$.)

Similarly, if we want this law to remain true for negative powers, we need

$$x^{-n} = \frac{1}{x^n}$$

For instance, $2^2 = 4$, and

$$\begin{aligned} 2^2 &= 2^{5-3} \\ &= 2^{5+(-3)} \\ &= 2^5 \cdot 2^{-3} \end{aligned}$$

or

$$2 \cdot 2 = (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) \cdot \left(2^{-3}\right)$$

Counting the 2s, we see we need

$$\begin{aligned} 2^{-3} &= \frac{1}{2^3} \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \end{aligned}$$

Warning! Taking powers is *not* associative:

$$(2^3)^2 = 8^2 = 8 \cdot 8 = 64$$

while

$$2^{(3^2)} = 2^9 = 512$$

So we need to be careful and use parentheses when appropriate.

Thinking about counting, we can get a simpler formula for $(x^n)^m$: This is m copies of x^n, and each of these copies contains n copies of x multiplied together. This means there are $m \cdot n$ total copies of x being multiplied together. In an equation,

$$(x^n)^m = x^{n \cdot m}$$

This formula also works for n, m negative or 0.

What if n is a fraction? It doesn't make sense to take $\frac{7}{4}$ copies of x and multiply them together, does it?

Remembering our basic law for how powers work, we have

$$\begin{aligned} x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} &= x^{(\frac{1}{2}+\frac{1}{2})} \\ &= x^1 \\ &= x \end{aligned}$$

This says $x^{\frac{1}{2}}$ is a number so that, when you multiply two copies together, you get x. This is commonly known as the 'square root' of x, also written $\sqrt{x}$

In general, $x^{\frac{1}{n}}$ is the n-th root of x, sometimes written $\sqrt[n]{x}$. If n is even, there are two possibilities:

$$2 \cdot 2 = 4 = (-2) \cdot (-2)$$

These two possibilites always come in a positive/negative pair. For us, $x^{\frac{1}{n}}$ when n is even is the **positive** n-th root, so

$$4^{\frac{1}{2}} = 2$$

Let's do a computation using both the $x^{\frac{1}{n}}$ notation and the $\sqrt[n]{x}$ notation:

$$\begin{aligned}(\sqrt{3}+\sqrt{2}) \cdot (\sqrt{3}-\sqrt{2}) &= \sqrt{3} \cdot \sqrt{3} + \sqrt{3} \cdot (-\sqrt{2}) + \sqrt{2} \cdot \sqrt{3} + \sqrt{2} \cdot (-\sqrt{2}) \\ &= 3 - \sqrt{3} \cdot \sqrt{2} + \sqrt{2} \cdot \sqrt{3} - 2 \\ &= 3 - 2 \\ &= 1\end{aligned}$$

$$\begin{aligned}(3^{\frac{1}{2}} + 2^{\frac{1}{2}}) \cdot ((3^{\frac{1}{2}} - 2^{\frac{1}{2}}) &= 3^{\frac{1}{2}} \cdot 3^{\frac{1}{2}} + 3^{\frac{1}{2}} \cdot (-2^{\frac{1}{2}}) + 2^{\frac{1}{2}} \cdot 3^{\frac{1}{2}} + 2^{\frac{1}{2}} \cdot (-2^{\frac{1}{2}}) \\ &= 3^{\frac{1}{2}+\frac{1}{2}} + 3^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} - 2^{\frac{1}{2}} \cdot 3^{\frac{1}{2}} - 2^{\frac{1}{2}+\frac{1}{2}} \\ &= 3 + 3^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} - 2^{\frac{1}{2}} \cdot 3^{\frac{1}{2}} - 2 \\ &= 3 - 2 \\ &= 1\end{aligned}$$

If we wanted, we could have combined $\sqrt{3} \cdot \sqrt{2} = \sqrt{6}$, which is the same as combining $3^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} = 6^{\frac{1}{2}}$. Since they cancel anyway, I did not do this.

Combining our above remarks, since $\frac{m}{n} = m \cdot \frac{1}{n}$ we have

$$\begin{aligned}x^{\frac{m}{n}} &= (x^m)^{\frac{1}{n}} \\ &= \sqrt[n]{x^m} \\ &= \left(\sqrt[n]{x}\right)^m\end{aligned}$$

These are all different ways of writing the same thing. In other words, when you raise a number to a fractional power, that is the same as taking the 'denominator'-th root and raising that to the 'numerator'-th power (or the other way around, you get the same number no matter which order). For instance

$$\begin{aligned} 8^{\frac{2}{3}} &= \left(8^{\frac{1}{3}}\right)^2 \\ &= 2^2 \\ &= 4 \end{aligned}$$

or

$$\begin{aligned} 8^{\frac{2}{3}} &= \left(8^2\right)^{\frac{1}{3}} \\ &= 64^{\frac{1}{3}} \\ &= 4 \end{aligned}$$

Thinking about how fractions multiply, we see that

$$\begin{aligned} \left(\frac{3}{8}\right)^3 &= \frac{3}{8} \cdot \frac{3}{8} \cdot \frac{3}{8} \\ &= \frac{27}{512} \\ &= \frac{3^3}{8^3} \end{aligned}$$

and in general

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

This is true even if n is a fraction.
For instance,

$$\begin{aligned} \left(\frac{16x^4}{25}\right)^{\frac{3}{2}} &= \left(\sqrt{\frac{16x^4}{25}}\right)^3 \\ &= \left(\frac{4x^2}{5}\right)^3 \\ &= \frac{64x^6}{125} \end{aligned}$$